Form-finding of Tensegrity Structures
テンセグリティ構造の形状決定

Jingyao ZHANG （張景耀）

Dept. of Design & Architecture

Nagoya City University, Japan

名古屋市立大学
Nagoya City University

January 13, 2015

Contents 内容

- Concept of Tensegrity 概念
- Applications 応用
- Stability 安定性
- Form-finding 形状決定

Key Problems for Preliminary Design

  - Intuition Approaches 直観的方法
  - Analytical Approaches 解析的方法
  - Numerical Approaches 数値的方法
    - Adaptive Force Density Method 適応軸力密度法
    - Dynamic Relaxation Method 動的緩和法
    - Non-linear Analysis Method 非線形解析法
    - Optimization Method 最適化手法

Tensegrity & R.B. Fuller

Tensegrity = **Tens**ional + Int**egrity** (R.B. Fuller, 1975)

The Montreal Biosphere
(Geodesic Dome, 1967)

Former pavilion of the United States
for the 1967 World Fair Expo

http://24.media.tumblr.com/tumblr_m1y6hqAlHg1qbajhlo1_400.jpg

R.B. Fuller
(1895-1983)

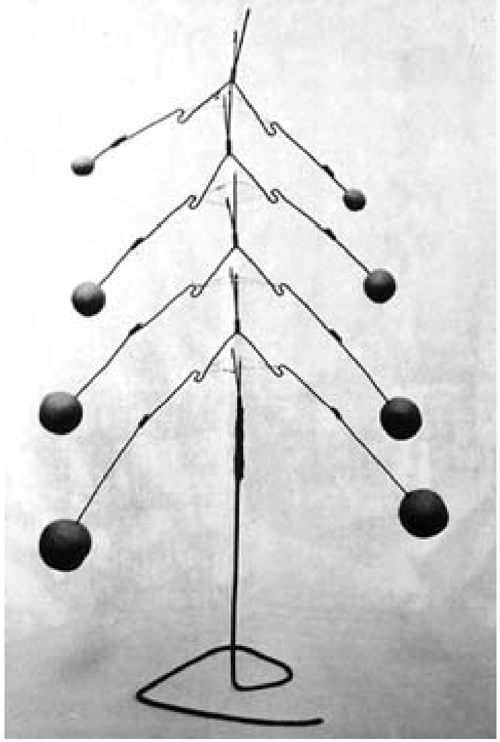
Question Posted by Fuller

Black Mountain College, USA in 1948

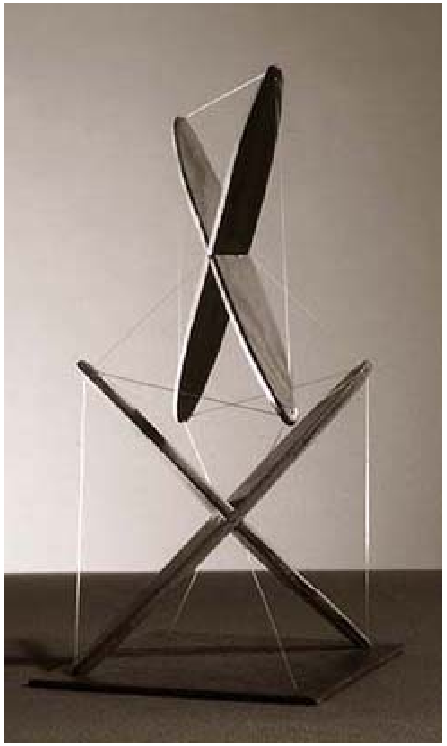
Question posted by Fuller to Snelson:

Is it possible to build a structure to illustrate the structural principle of nature, which was observed to rely on that continuous tension embraces isolated compression elements?

"islands of compression reside in a sea of tension"



Spinal-column
(1948)

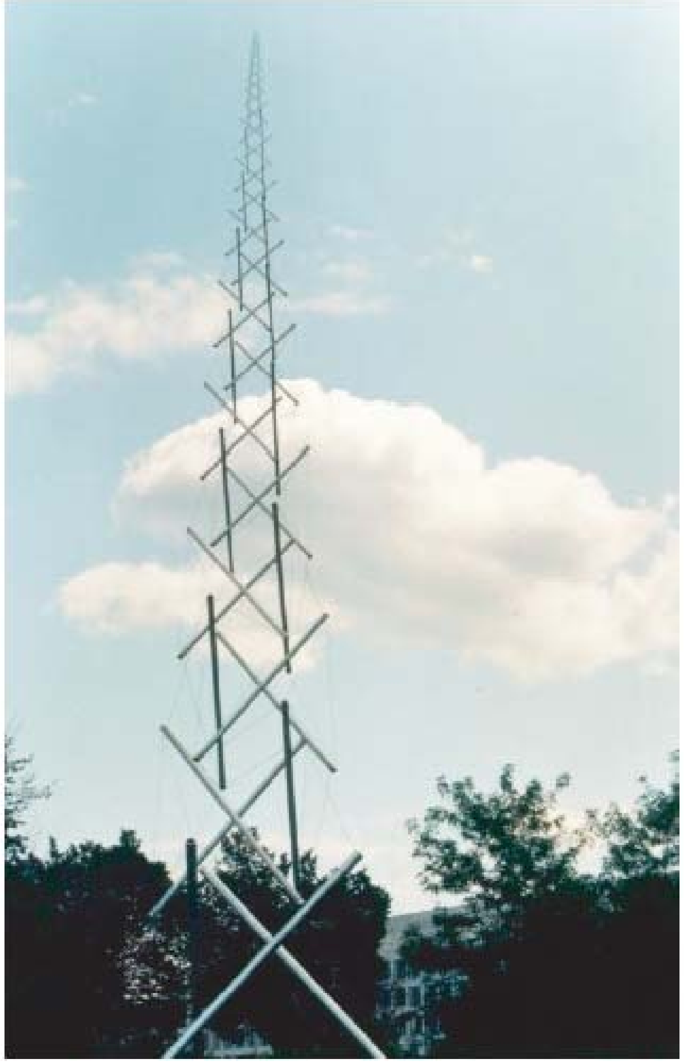


X-column
(1948)

R. B. Fuller

K. Snelson

Tensegrity by Snelson



Needle Tower

(18m, 1968)

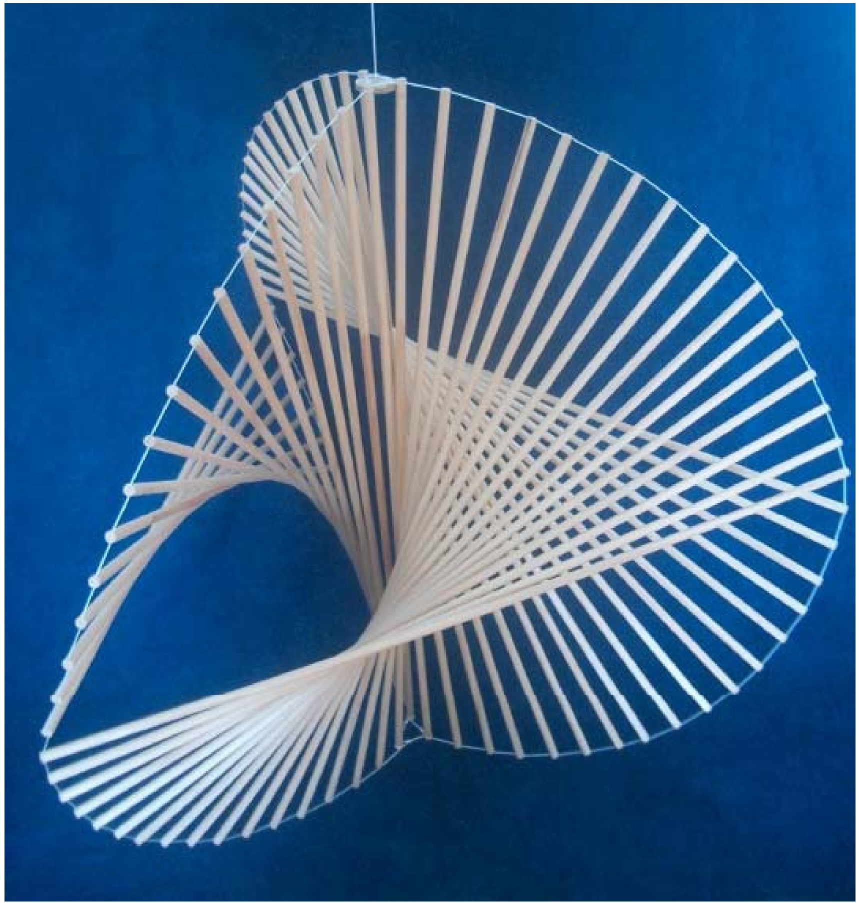
Tensegrity by Snelson

Triple Crown, 1991
13 x 26 x 23m

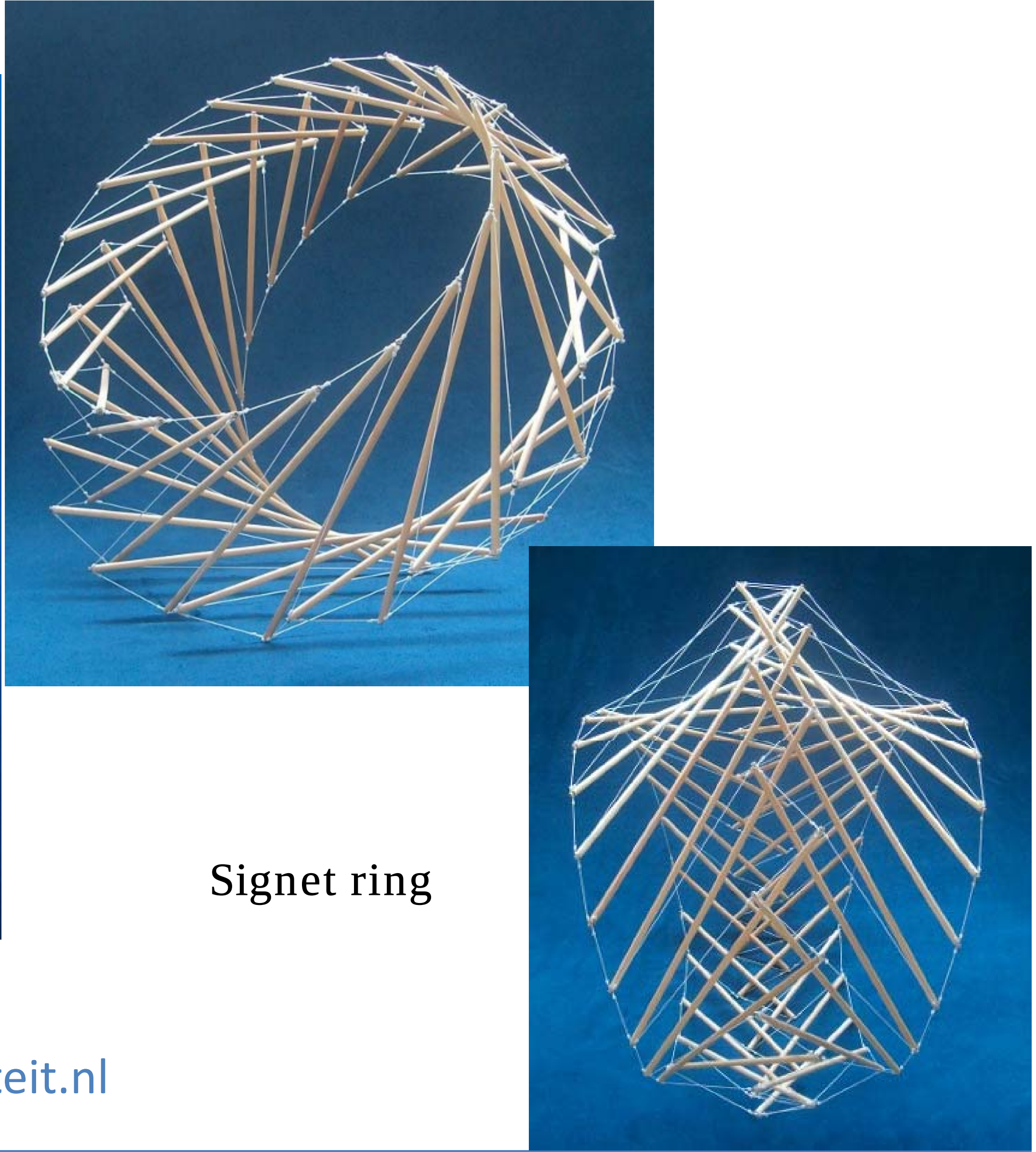
http://kennethsnelson.net

Sleeping Dragon, 2002
3 x 22.1 x 4.8m

Tensegrity Art works



Three Fans

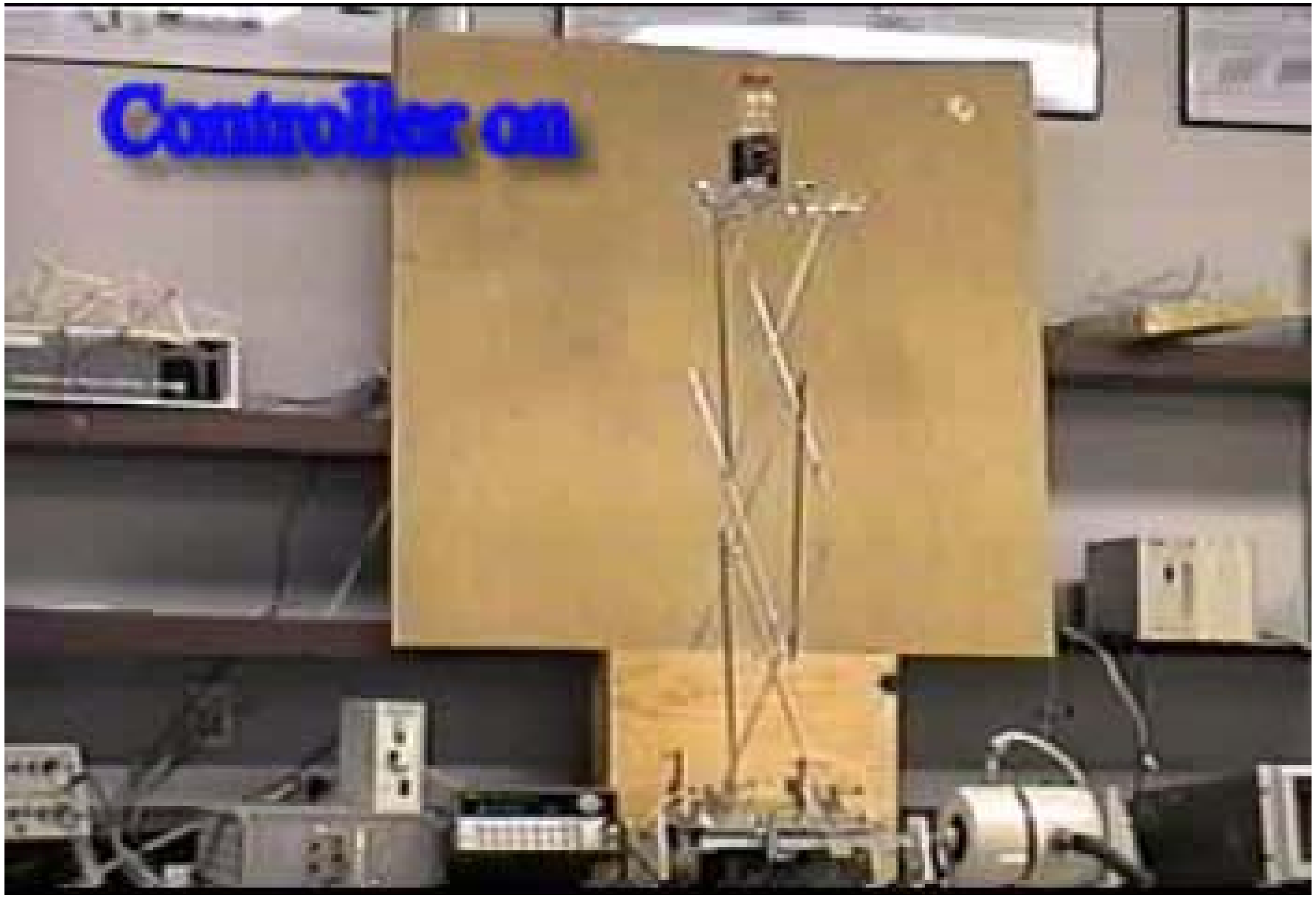


Signet ring

http://www.tensegriteit.nl

Student Works

Applications: Vibration Control 振動制御



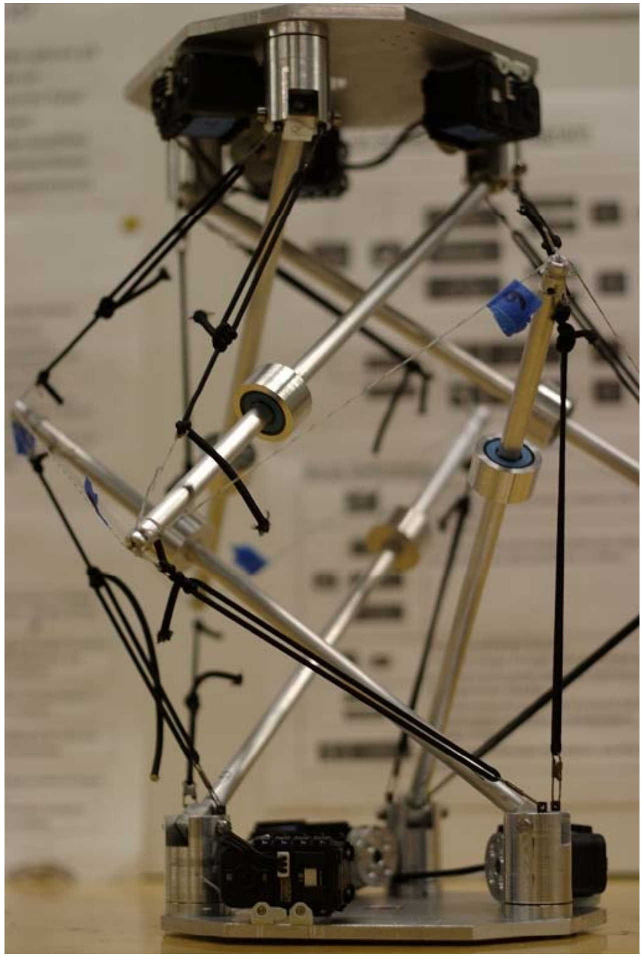
Structural Systems and Control Laboratory
Department of Mechanical and Aerospace Engineering
University of California, San Diego

http://maeresearch.ucsd.edu/skelton/laboratory/vibration_control.htm

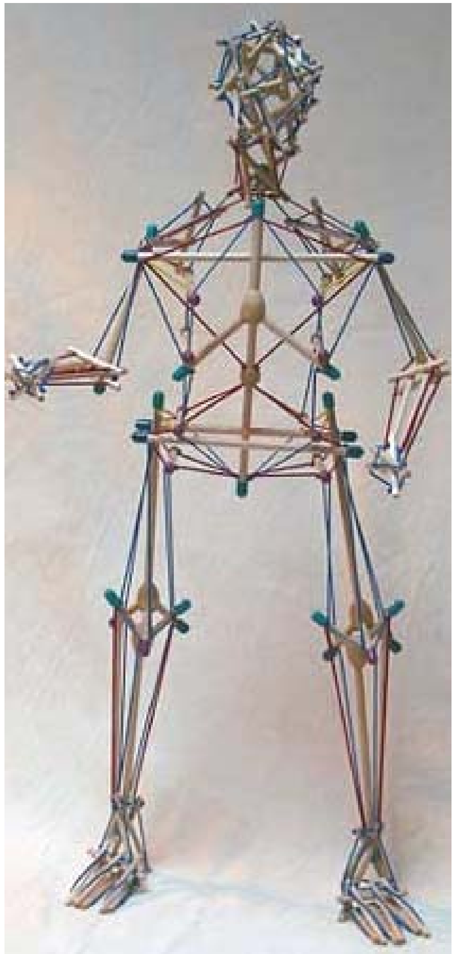
Applications: Deployable　展開構造

Deployable Structure, G. Tibert (2002)

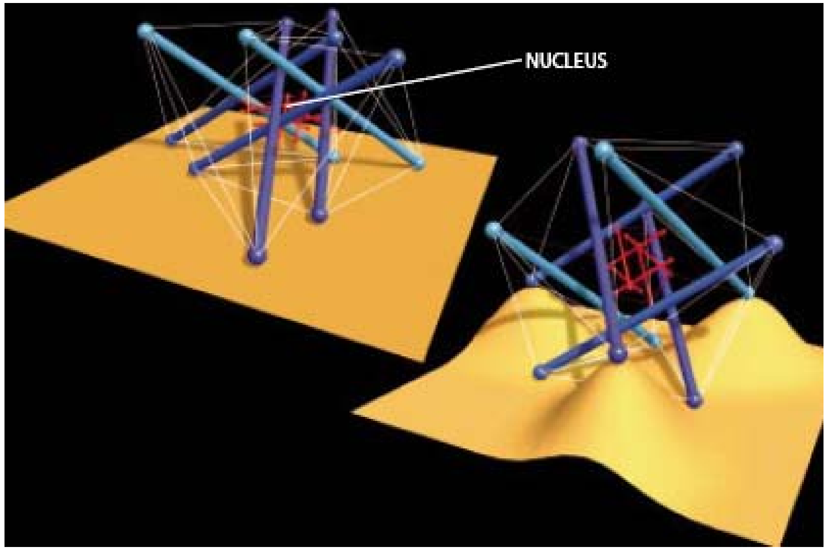
Applications



Robot



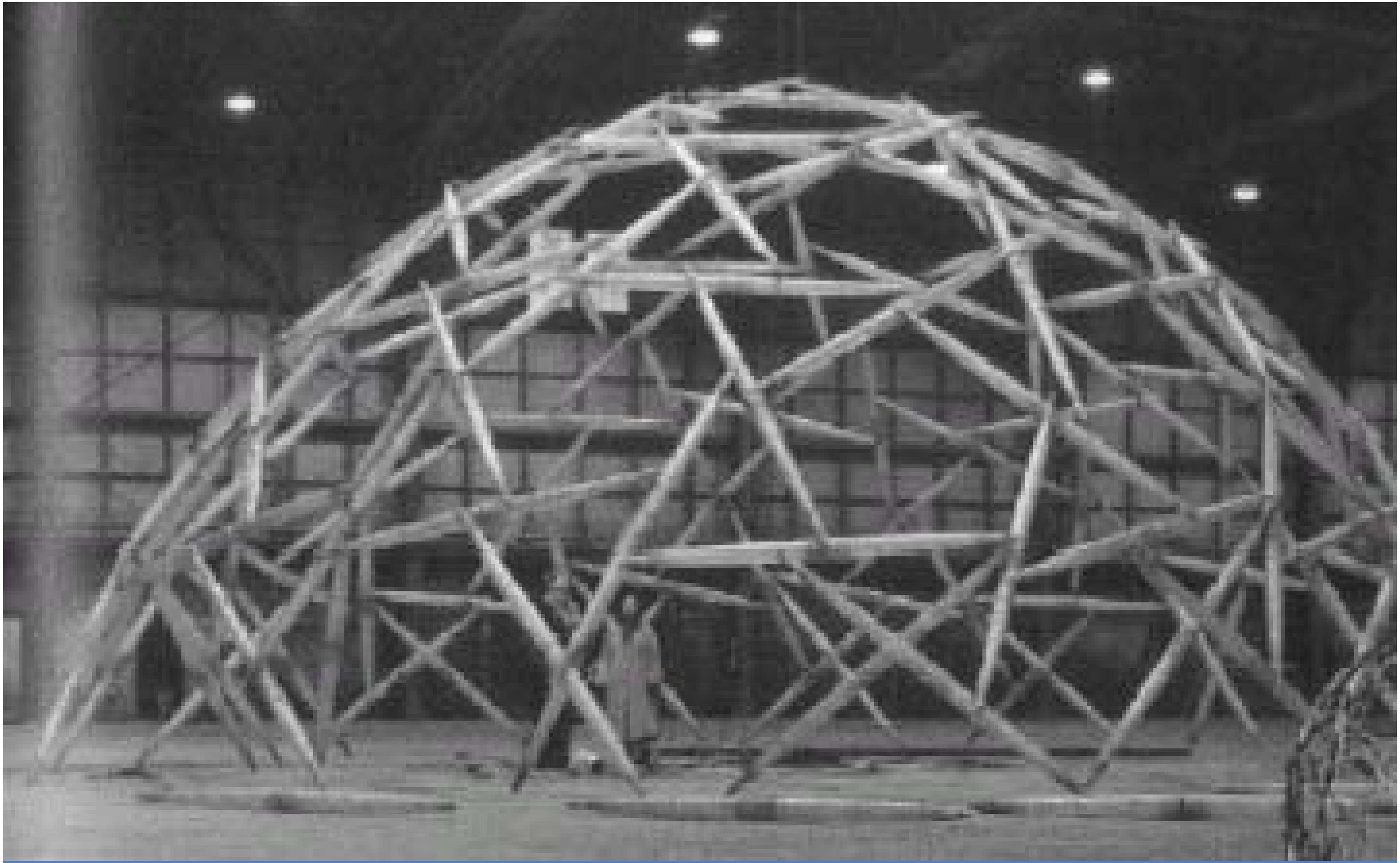
Human Skelton



Morphology of Cells

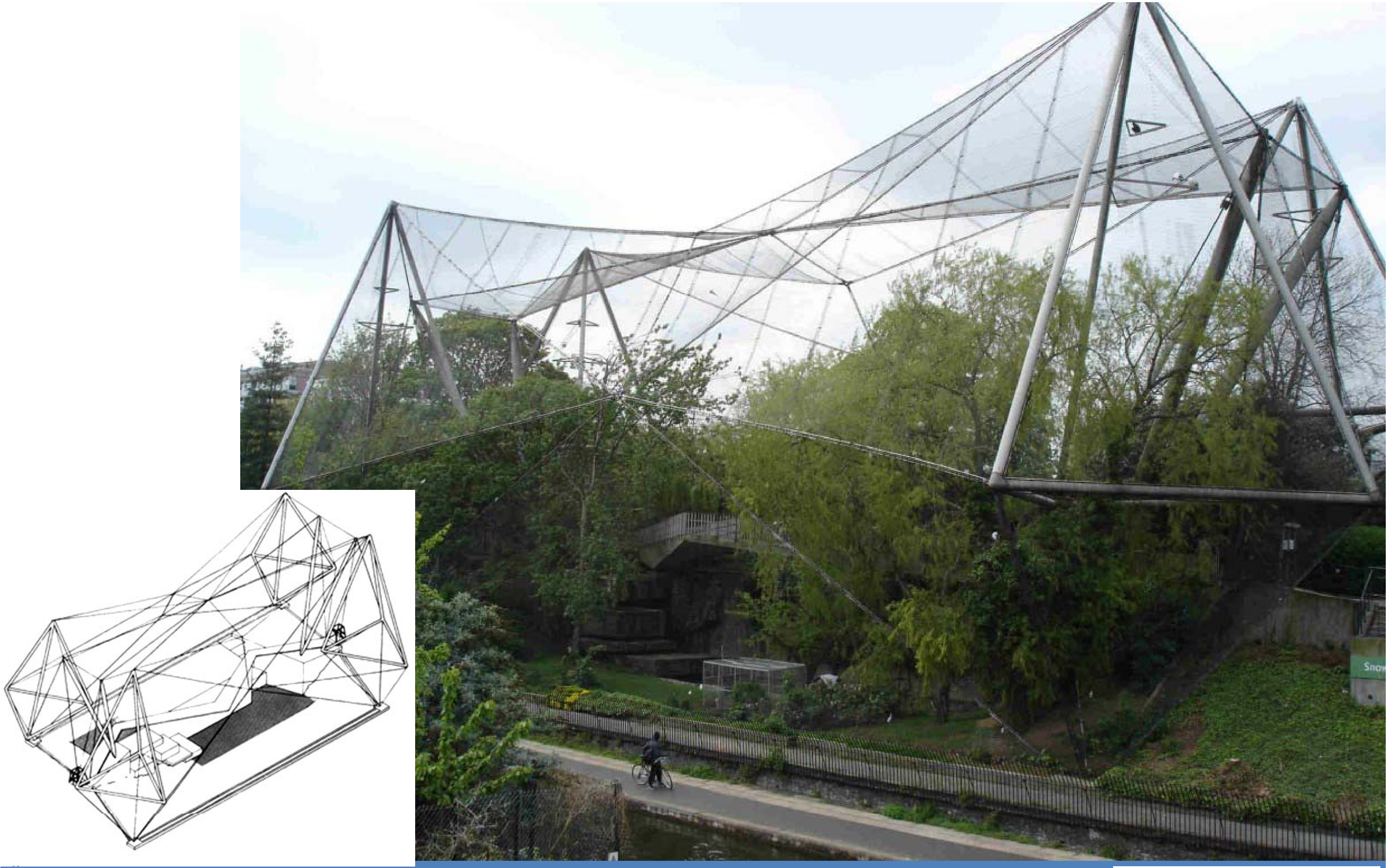
etc. . .

Tensegrity Dome (1953)



Aviary @ London Zoo (1964)

Designed by Lord Snowdon in 1964



Experimental Facility (2001)

Experimental Facility at Tokyo University, Designed by K. Kawaguchi

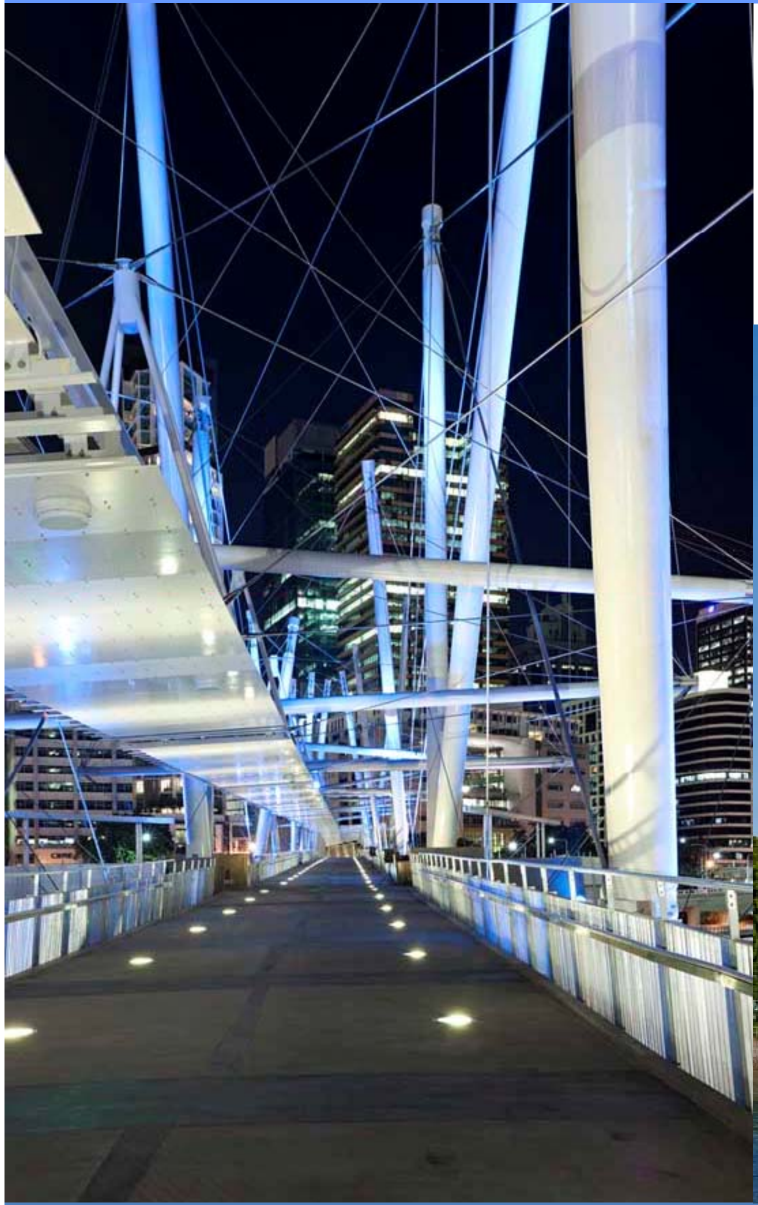
Kurilpa Bridge (2009)

Designed by the Queensland division of
Australia’s Cox Rayner Architects with Arup

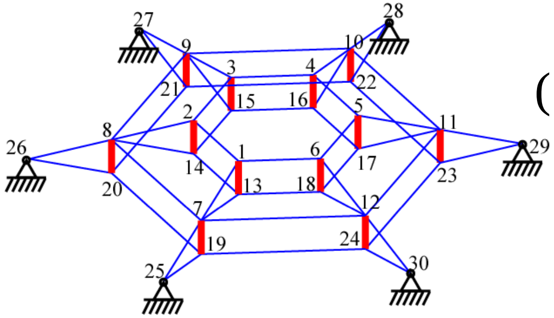
Pedestrian and Cycle Bridge

470 meters long and 6.5 meters wide

http://www.e-architect.co.uk/brisbane/kurilpa-bridge



Cable-Dome ケーブルドーム



Kano Dome
(1991, Shizuoka, Japan)

○ Light-weight
- Higher Stiffness
- Local compression
  ⇒ High buckling load
- Global tension
  ⇒ High-strength material

○ Simple joint

○ Strong to earthquake

△ Sensitive to wind

△ Difficult in prestress

Others: Gymnastic and fencing stadiums Seoul, 1988, South Korea
Georgia Dome, 1992, Atalanta, USA

Contents

- Introduction to Tensegrity
- Applications
- Stability
- <u>Form-finding</u>
    - Intuition Approaches
    - Analytical Approaches (using symmetry)
    - <u>Numerical Approaches</u>
        - Adaptive Force Density Method
        - Dynamic Relaxation Method
        - Non-linear Analysis Method
        - Optimization Method

Assumptions & Definitions 仮定・定義

Preliminary Design Problem BEFORE

- Selection of real materials
- Structural design subjected to real loads

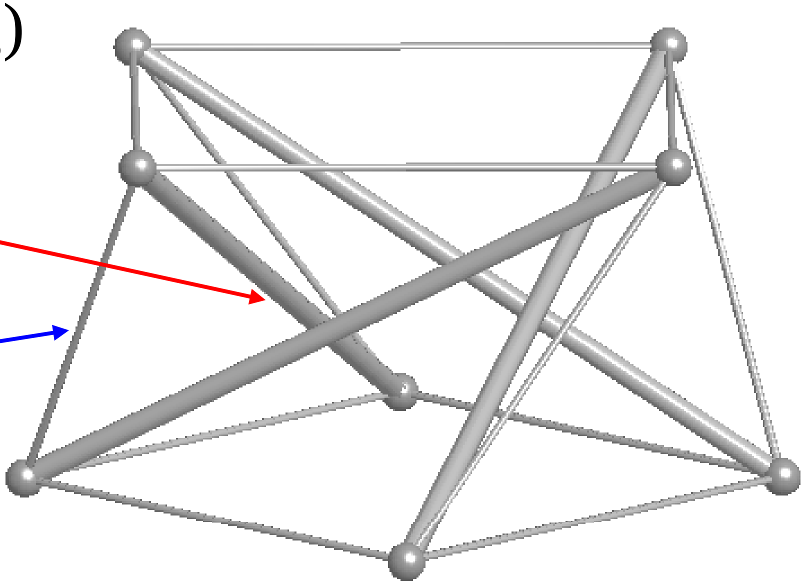
Mechanical Assumptions

- Pin-jointed
- No self-weight nor external load
- No member failure (yielding nor buckling)
- No support (free-standing)

Member Definitions

- Strut (in compression)

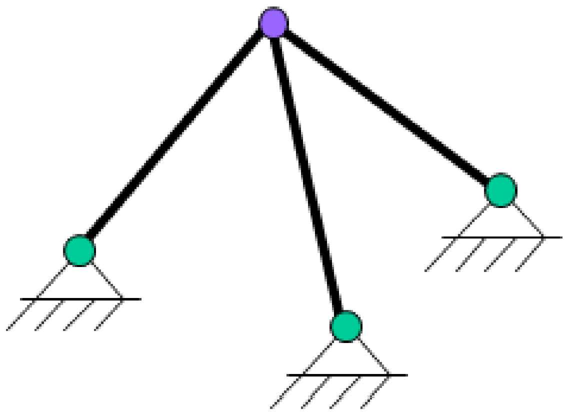
- Cable (in tension)



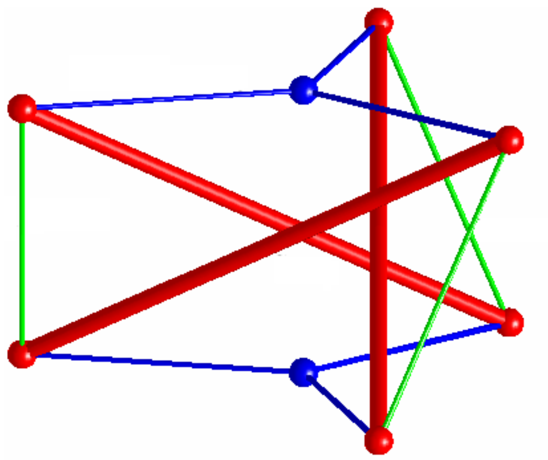
Maxwell's Rule & Exception　マックスウェルルールと例外

Maxwell's Rule Constraints + Members – Nodes x Dimension

>= 0



3x3 + 3 – 4x3 = 0 → Stable



12 + (6) – 8x3 = – 6 < 0 → Unstable?

Rigid-body motions

One self-equilibrium force mode
Seven infinitesimal mechanisms

Stability 安定性

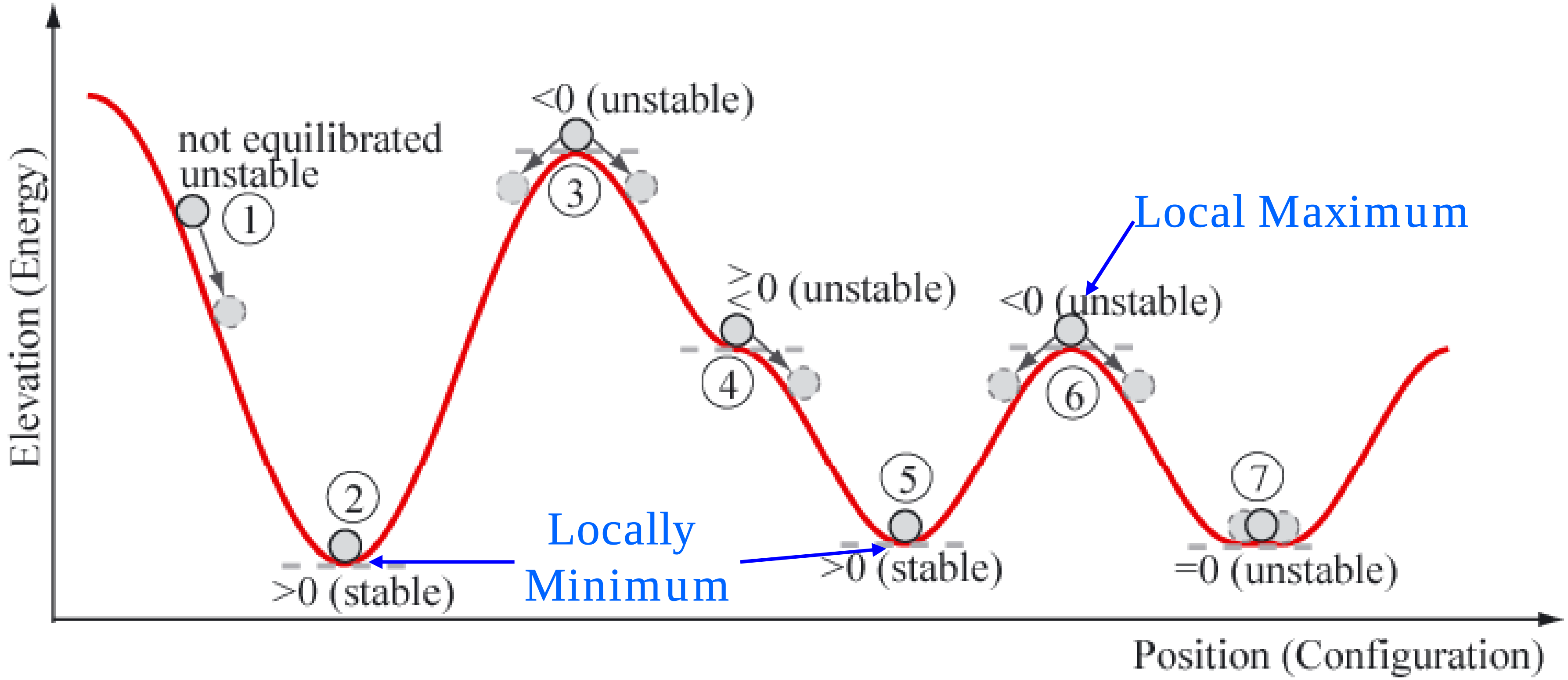
Tangent Stiffness: $\mathbf{K} = \mathbf{K}_E + \mathbf{K}_G$

Prestresses

Shape & Material

Stability: $\mathbf{K}$ P.D. ⇨ $\mathbf{d}^{\mathrm{T}}\mathbf{K}\mathbf{d} > 0$ ← Positive increase of strain energy

Super-stability: $\mathbf{K}_G$ P.S.D. ← Non-negative eigenvalues

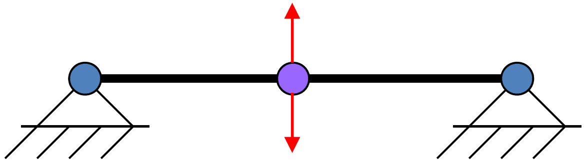
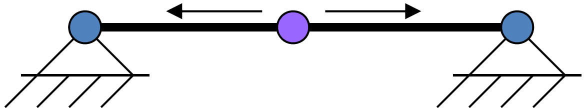
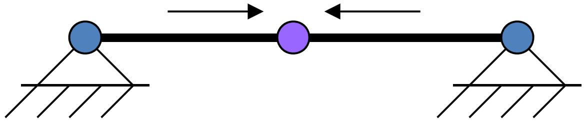


Super-stability

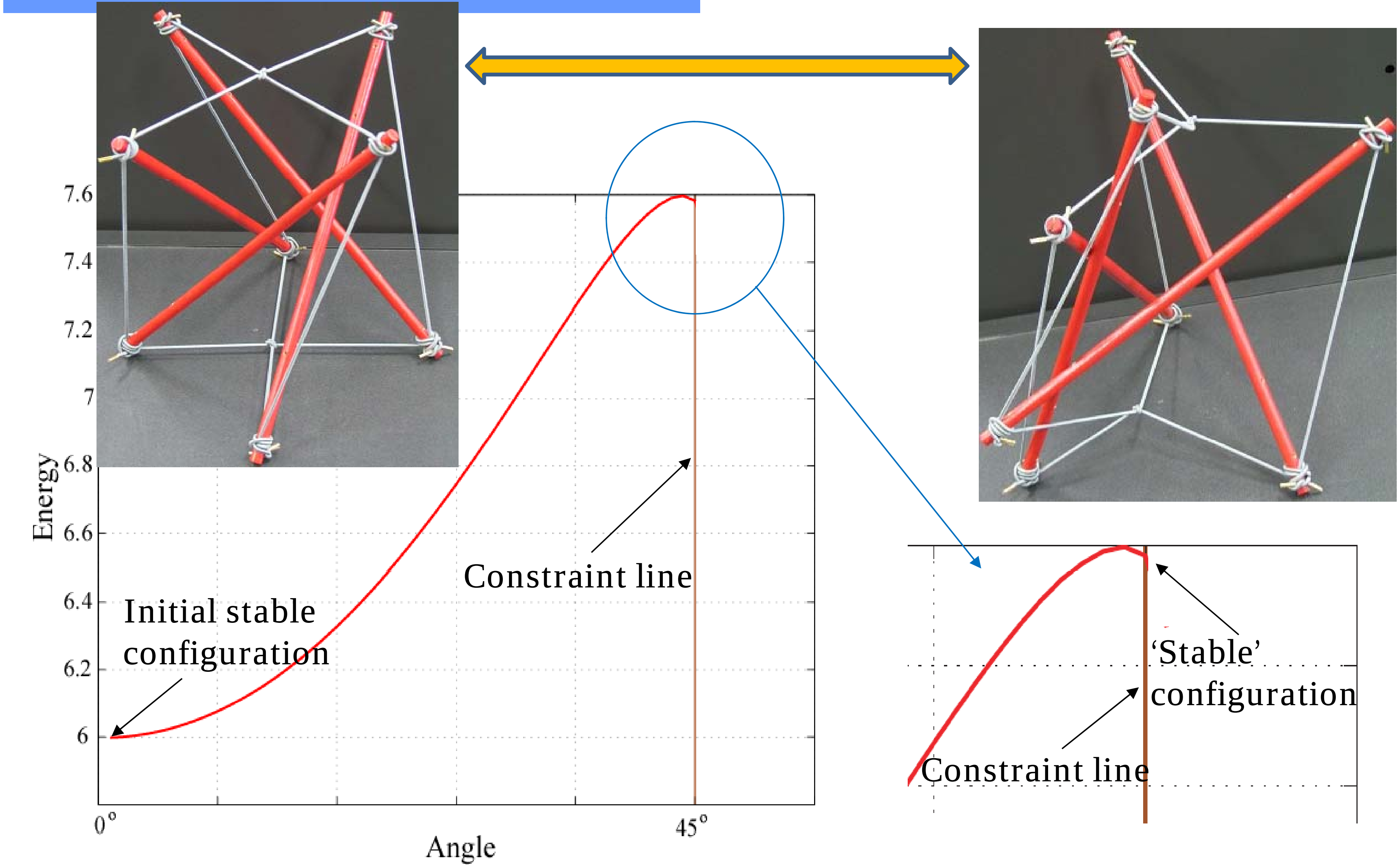
Shape & Material

Tangent Stiffness: $\mathbf{K} = \mathbf{K}_E + \mathbf{K}_G$

Prestresses

No prestress	Mechanism  Neutrally Stable	$\mathbf{K}_G = 0$ $\mathbf{K}_E \ge 0$ $\mathbf{K} \ge 0$
Tension Positive force	 Super-stable	$\mathbf{K}_G > 0$ $\mathbf{K}_E \ge 0$ $\mathbf{K} > 0$
Compression Negative force	 Unstable	$\mathbf{K}_G < 0$ $\mathbf{K}_E \ge 0$ $\mathbf{K} < 0$

Multi-stable Tensegrity



Contents

- Introduction to Tensegrity
- Applications
- Stability
- Form-finding (or Shape-finding)
  - Intuition Approaches
  - Analytical Approaches (using symmetry)
  - Numerical Approaches
    - Adaptive Force Density Method
    - Dynamic Relaxation Method
    - Non-linear Analysis Method
    - Optimization Method

Configuration & Prestresses
at the state of Self-equilibrium

Form-finding of Arch

$$y = a\cosh\left(\frac{x}{a}\right) = \frac{a}{2}\left(e^{x/a} + e^{-x/a}\right)$$

Arch
(compression)

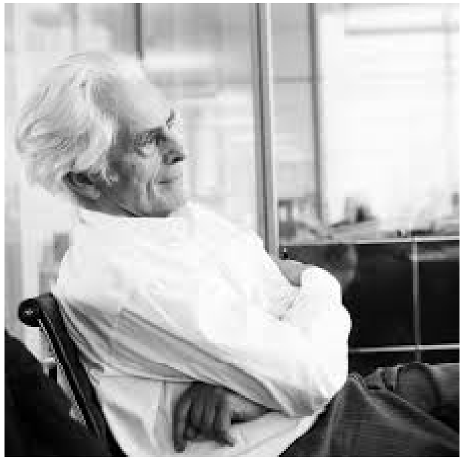
Catenary
(tension)

Form-finding of Basilica and Expiatory Church of the Holy Family

Constructed from 1882
UNESCO World Heritage Site in 2005

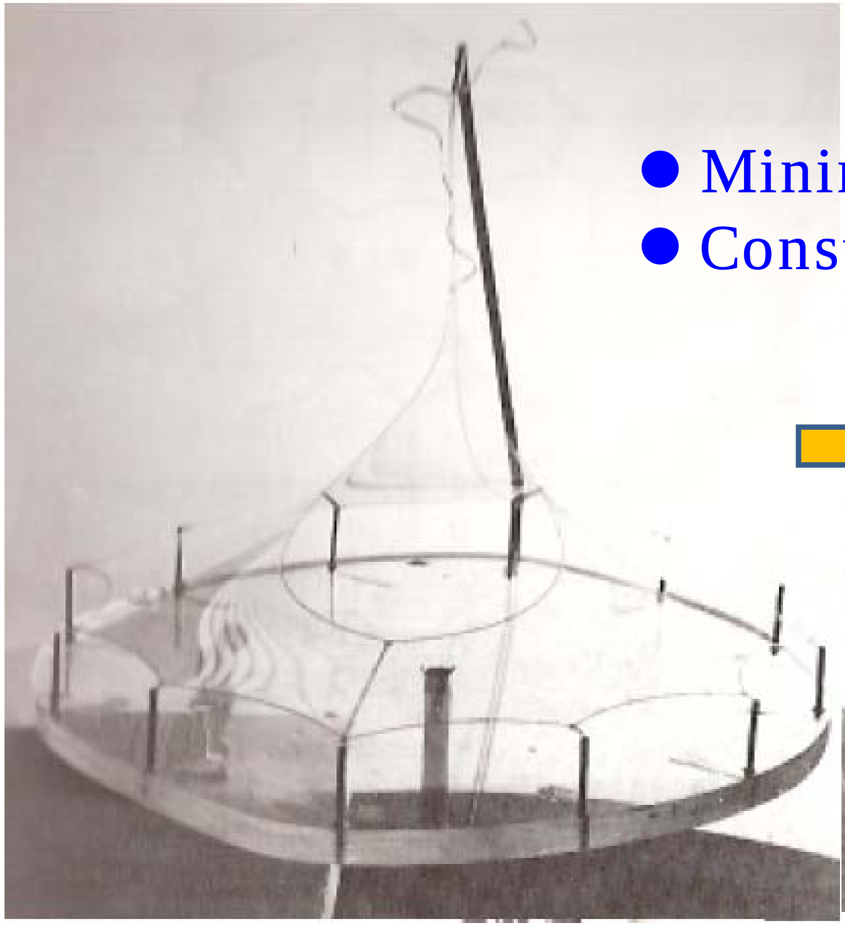
Sand bag model to find the natural shape (minimum bending moment)

Form-finding of Cable-net
ケーブルネットの形状決定



Frei Otto
(1925~)

- Minimal area
- Constant stress



Soap-bubble Experiment

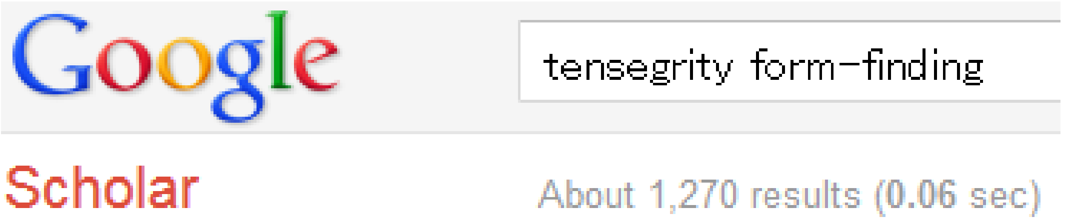
Cable-net

West German Pavilion (1967)

1967 World Fair Expo
in Montreal, Canada

Contents

- Introduction to Tensegrity
- Applications
- Stability
- Form-finding
  - Intuition Approaches
  - Analytical Approaches (using symmetry)
  - Numerical Approaches
    - Adaptive Force Density Method
    - Dynamic Relaxation Method
    - Non-linear Analysis Method
    - Optimization Method

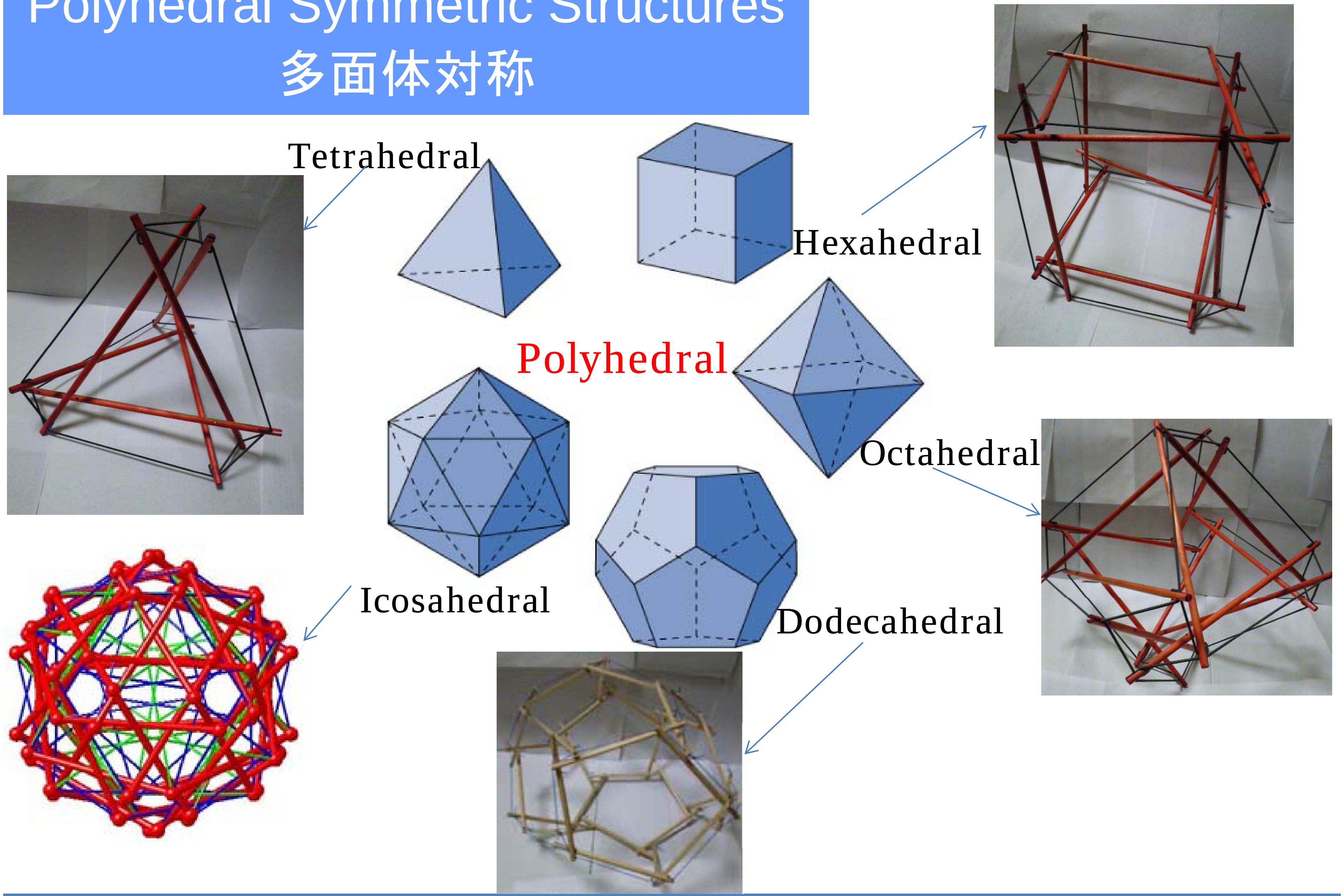


Google Scholar

tensegrity form-finding

About 1,270 results (0.06 sec)

Polyhedral Symmetric Structures
多面体対称



Tetrahedral

Hexahedral

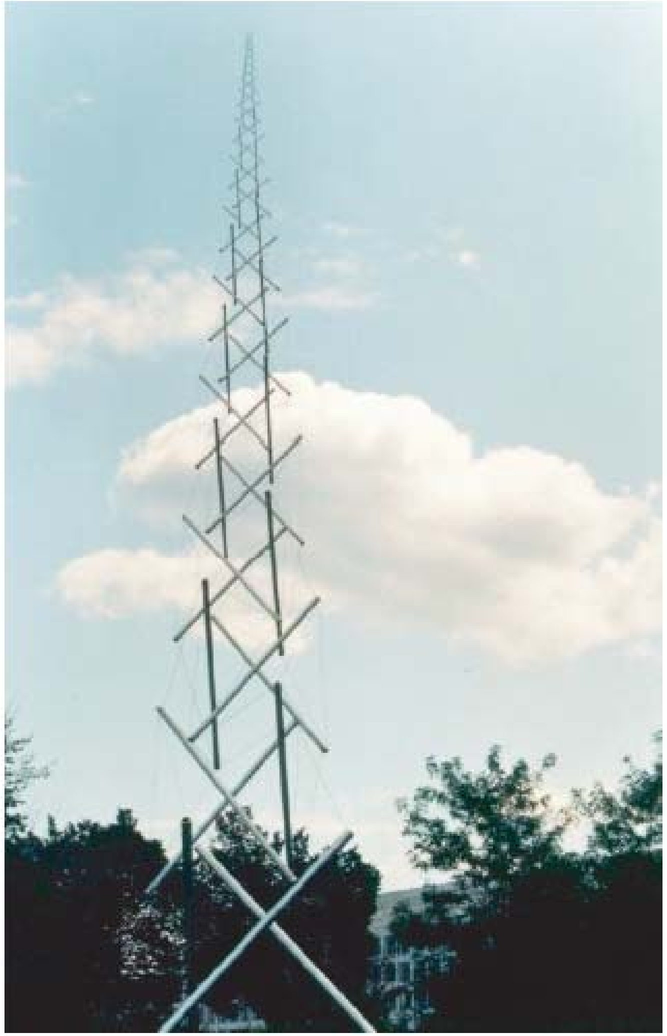
Polyhedral

Octahedral

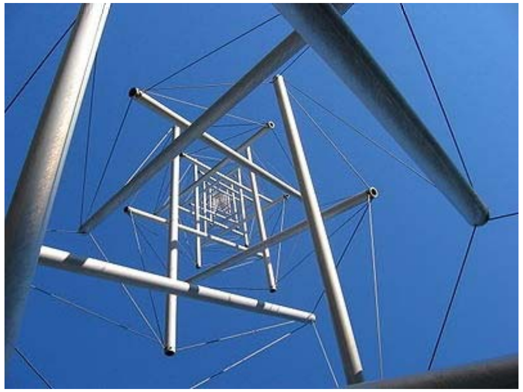
Icosahedral

Dodecahedral

Tensegrity Tower テンセグリティ･タワー



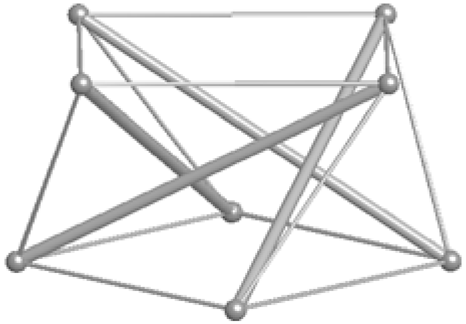
Hirshhorn Museum & Sculpture Garden, Washington, D.C.



Needle Tower (18m, 1968)

Needle Tower II (30m, 1969)

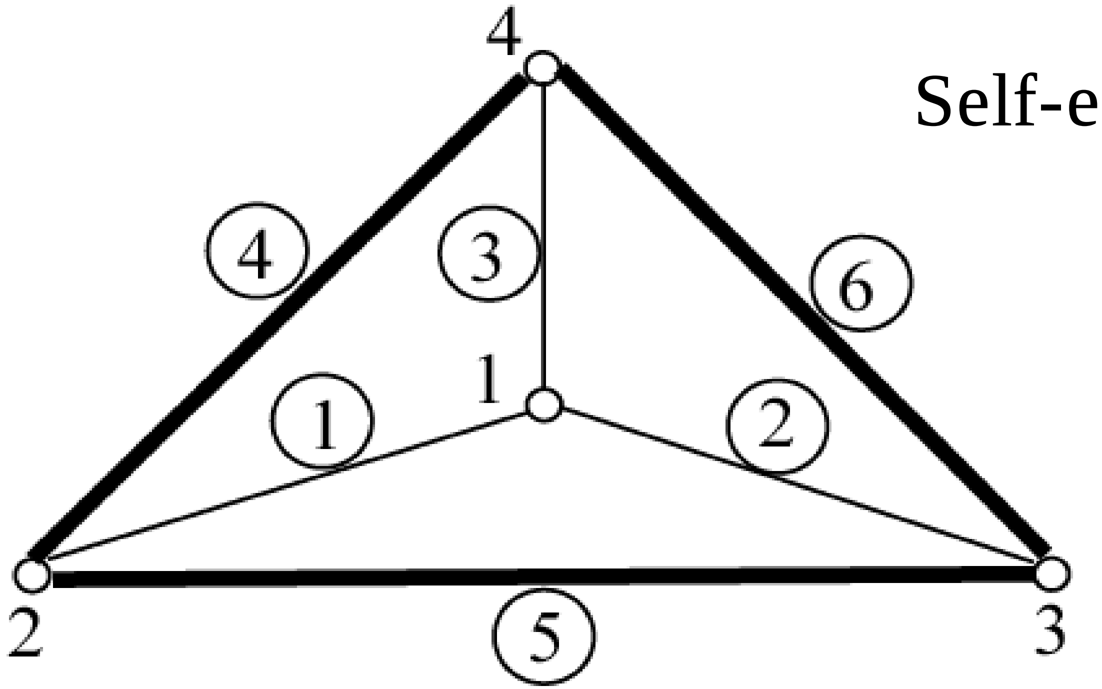
by K. Snelson



Unit cell

Kröller Müller Museum, Otterlo, Holland

Self-equilibrium Equations　自己釣合い方程式



Self-equilibrium Equations:

$$\mathbf{Ex} = \mathbf{Ey} = \mathbf{Ez} = \mathbf{0}$$

E:	Force density matrix
x,**y**,**z**:	Nodal coordinates

Force Density $q_k = f_k / l_k$

$$\mathbf{E} = \begin{pmatrix} q_1+q_2+q_3 & + & -q_1 & + & -q_2 & + & -q_3 = \mathbf{0} \\ & & q_1+q_4+q_5 & & -q_5 & & -q_4 \\ & & & & q_2+q_5+q_6 & & -q_6 \\ & \text{Sym.} & & & & & q_3+q_4+q_6 \end{pmatrix}$$

Self-equilibrium & Super-stability Conditions

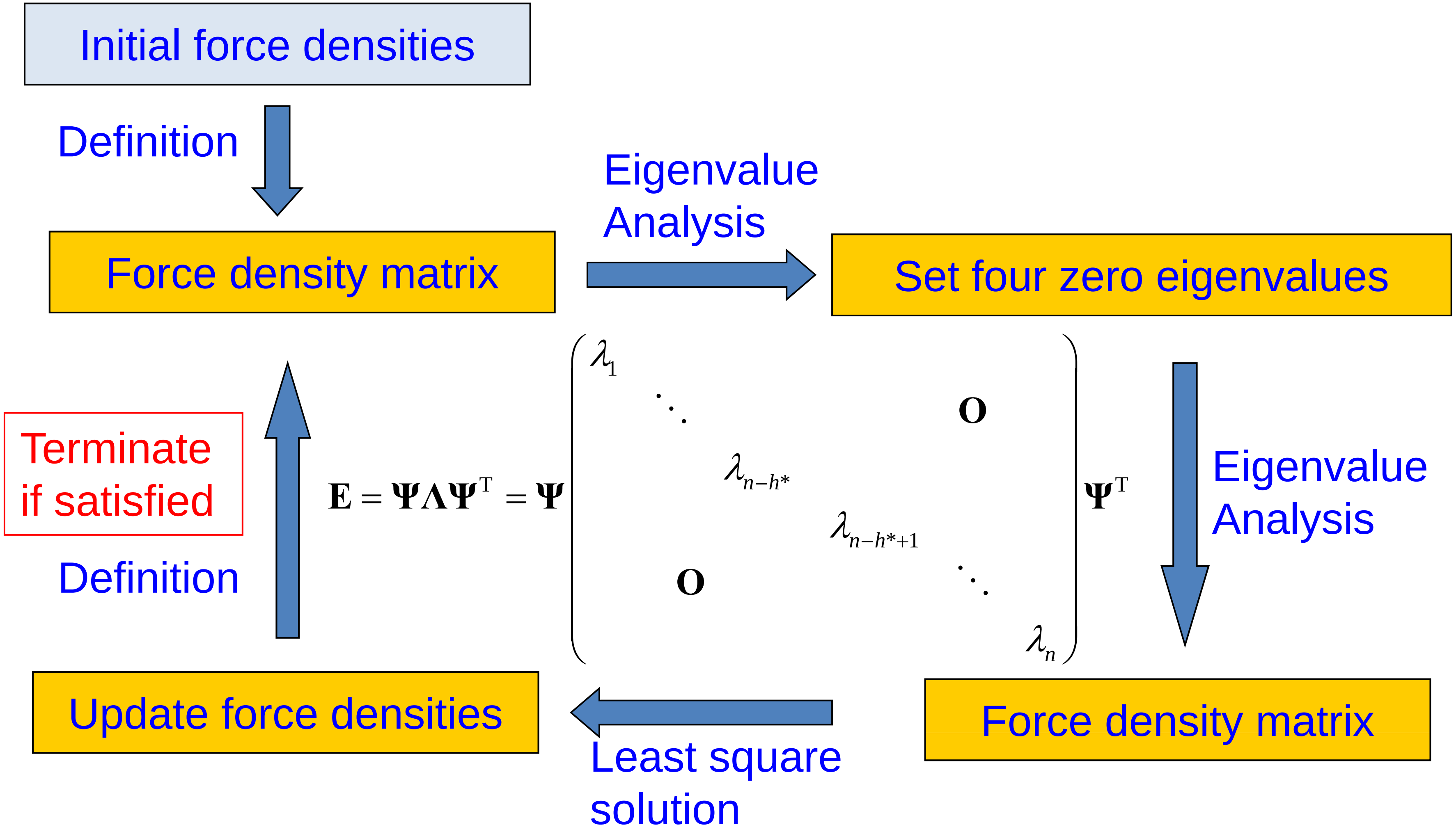
★ $$\mathbf{Ex} = \mathbf{0} \Longrightarrow \mathbf{x}, \mathbf{y}, \mathbf{z} = \alpha_1 \begin{pmatrix} 1 \\ \vdots \\ 1 \end{pmatrix} + \sum_{i=2}^{h} \alpha_i \mathbf{P}_i$$

- $h >= 4$ ⇒ Three-dimensional

★ $$\mathbf{K}_G = \begin{bmatrix} \mathbf{E} & & \\ & \mathbf{E} & \\ & & \mathbf{E} \end{bmatrix}$$ P.S.D. for super-stability

☆**E** has four 0 eigenvalues	Self-equilibrium
☆Other eigenvalues are positive	Super-stability

Adaptive Force Density Method (AFDM)適応軸力密度法



Initial force densities

Definition

Force density matrix

Eigenvalue Analysis

Set four zero eigenvalues

$$\mathbf{E} = \mathbf{\Psi}\mathbf{\Lambda}\mathbf{\Psi}^{\mathrm{T}} = \mathbf{\Psi}\begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & \mathbf{O} & \\ & & \lambda_{n-h^*} & & & \\ & & & \lambda_{n-h^*+1} & & \\ & \mathbf{O} & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix}\mathbf{\Psi}^{\mathrm{T}}$$

Eigenvalue Analysis

Force density matrix

Least square solution

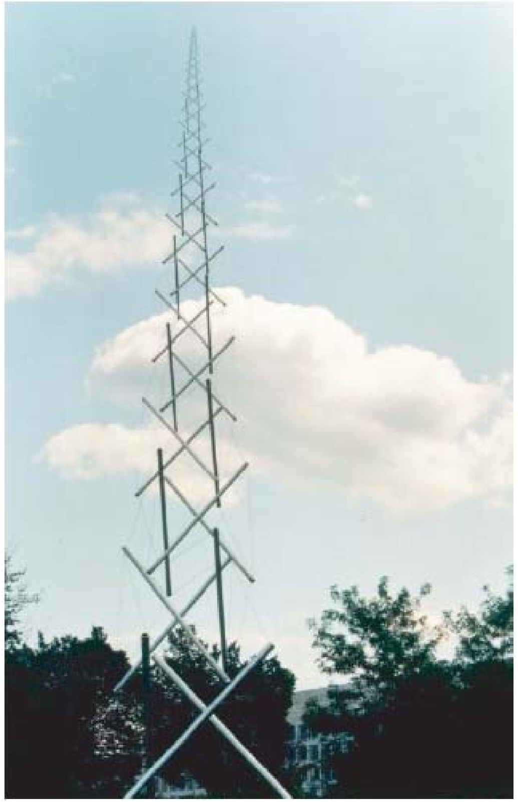
Update force densities

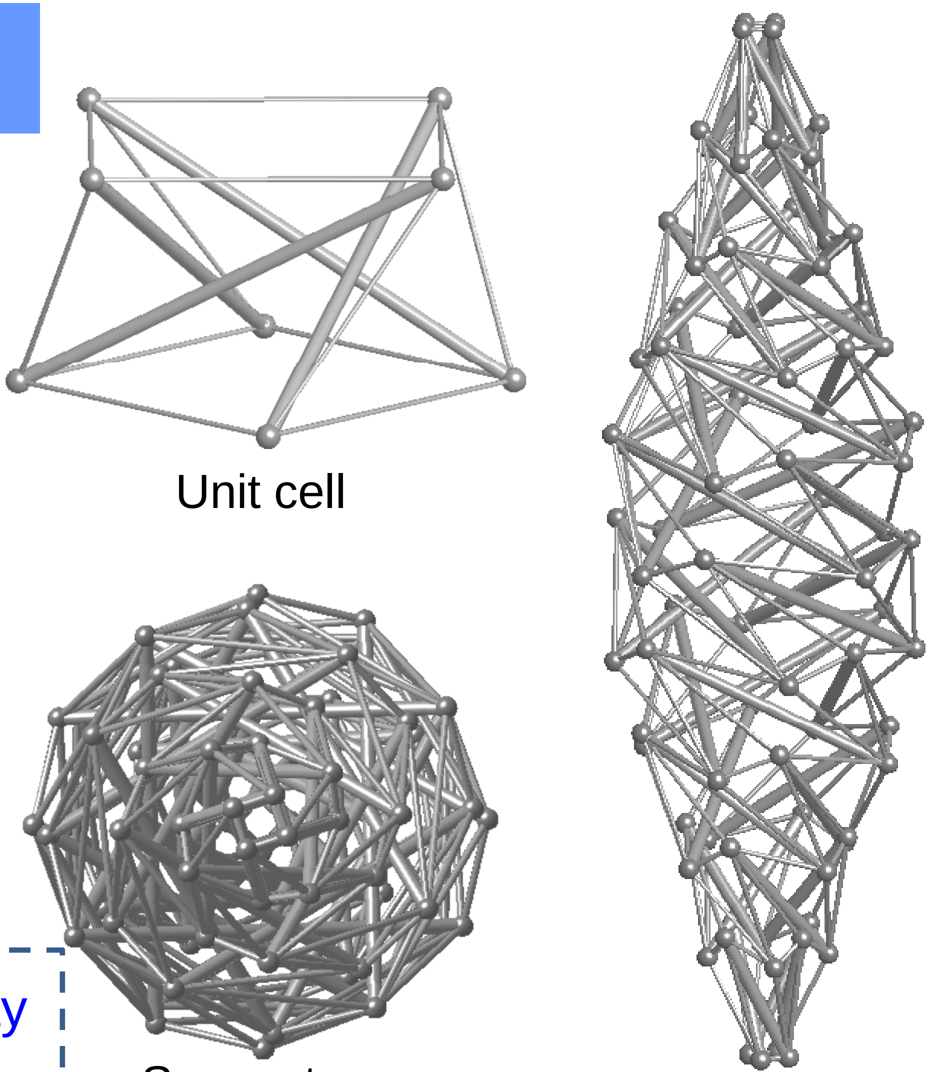
Definition

Terminate if satisfied

- J.Y. Zhang and M. Ohsaki
Form-finding of tensegrity structures subjected to geometrical constraints, *Int. J. Space Structures*, Vol. 21, 4, pp. 183-195, 2006.

Tensegrity Tower by AFDM





Unit cell

Symmetry
about *z*-axis

Specified
z-coordinates

Ten-story tower

Merit: Guarantee on super-stability
Low computational costs
Geometrical constraints

Demerit: Less accurate shape control

Contents

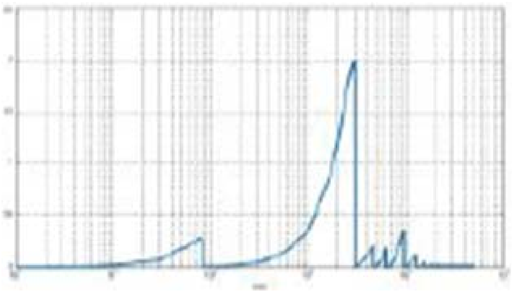
- Introduction to Tensegrity
- Applications
- Stability
- Form-finding
    - Intuition Approaches
    - Analytical Approaches (using symmetry)
    - Numerical Approaches
        - Adaptive Force Density Method
        - Dynamic Relaxation Method
        - Non-linear Analysis Method
        - Optimization Method

Dynamic Relaxation Method (DRM) 動的緩和法

Equilibrium Equation

$$\mathbf{Ds}=\mathbf{p}$$

D: Equilibrium matrix
s: Axial force vector
p: Out-of-balance force



Motion Equation

$$\mathbf{M}\ddot{\mathbf{x}}+\mathbf{C}\dot{\mathbf{x}}+\mathbf{K}\mathbf{x}=-\mathbf{p}$$

$$E=\mathbf{M}\dot{\mathbf{x}}^2/2$$

M: Mass matrix
C: Damping
K: Stiffness matrix
x: Displacement vector
E: Kinetic energy

Initial configuration — Out-of-balance

Motion Equation

New configuration — Out-of-balance

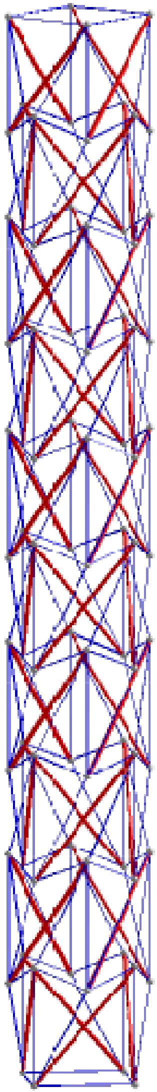
Motion Equation

Reset velocity to zero if kinetic energy reaches its peak value; Terminate if the peak value is small.

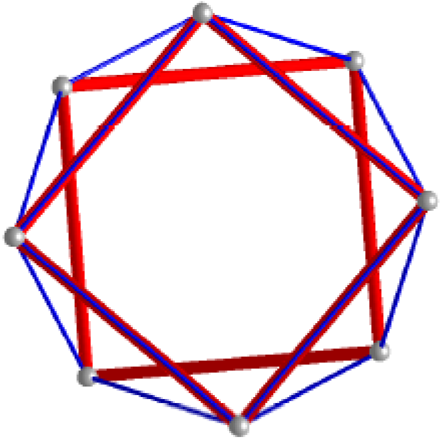
Final configuration — Self-equilibrated

- J.Y. Zhang and M. Ohsaki,
Free-form Design of Tensegrity Structures by Dynamic Relaxation Method, Proc. IASS-IACM 2012, Sarajevo, Bosnia and Herzegovina, April 2012.

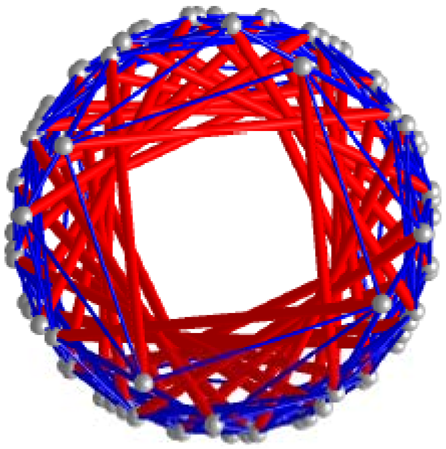
Tensegrity Tower by DRM



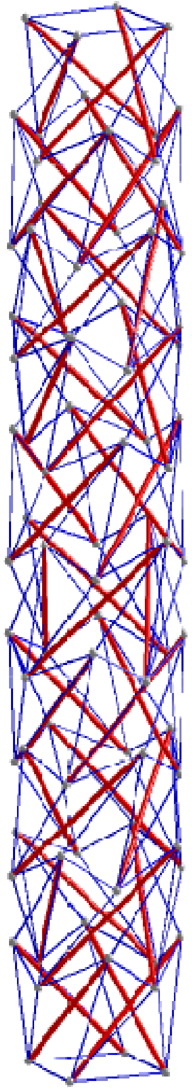
Initial



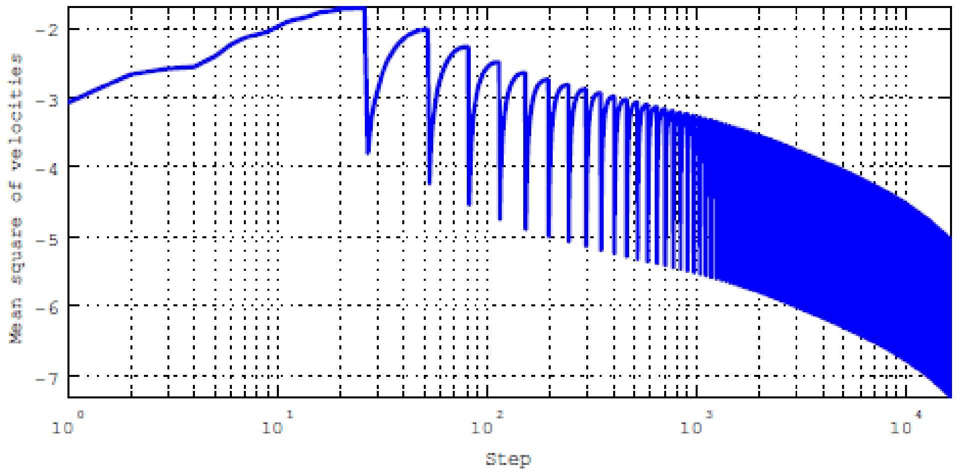
Initial



Final



Final



Kinetic energy

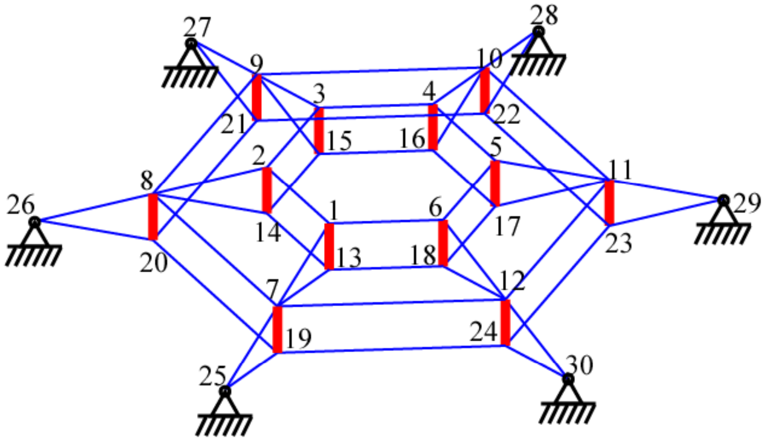
Merit: Close to designed shape

Demerit: Low convergence

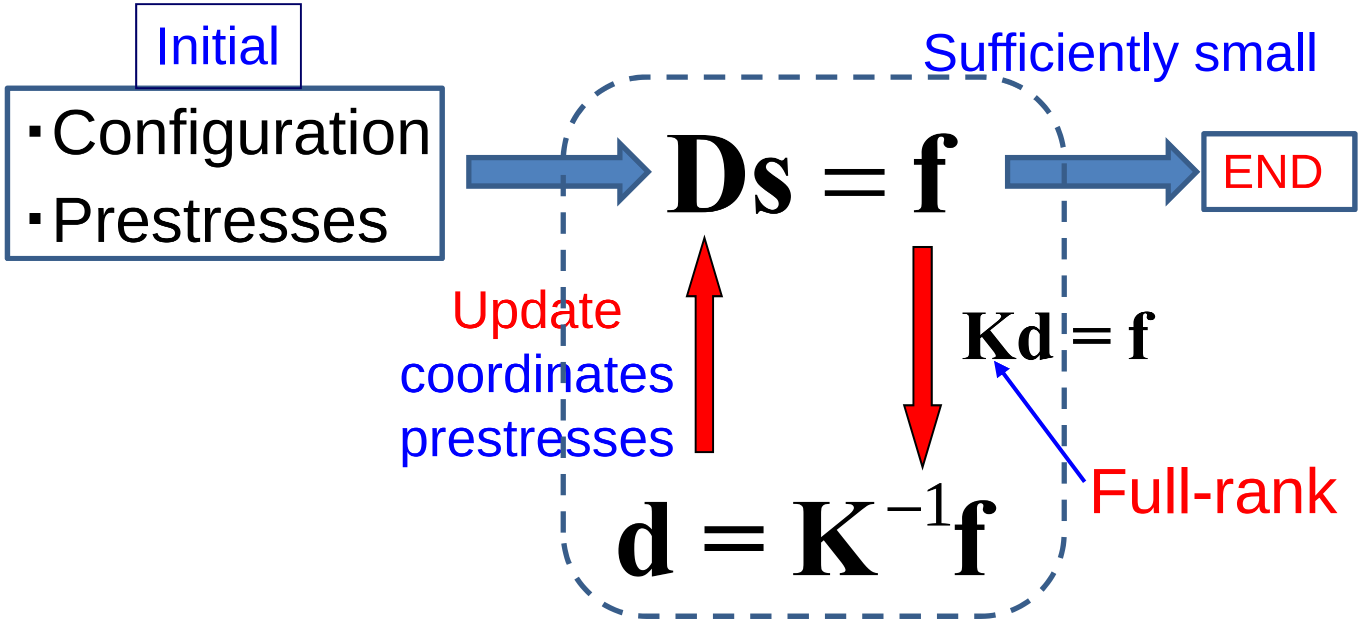
Contents

- Introduction to Tensegrity
- Applications
- Stability
- Form-finding
  - Intuition Approaches
  - Analytical Approaches (using symmetry)
  - Numerical Approaches
    - Adaptive Force Density Method
    - Dynamic Relaxation Method
    - Non-linear Analysis Method
    - Optimization Method

Non-linear Analysis (NLA) 非線形解析法



D: Equilibrium matrix
s: Prestress vector
f: Unbalanced force vector
K: Stiffness matrix
d: Nodal coordinates



Initial

- Configuration
- Prestresses

$$\mathbf{Ds} = \mathbf{f}$$

Sufficiently small

END

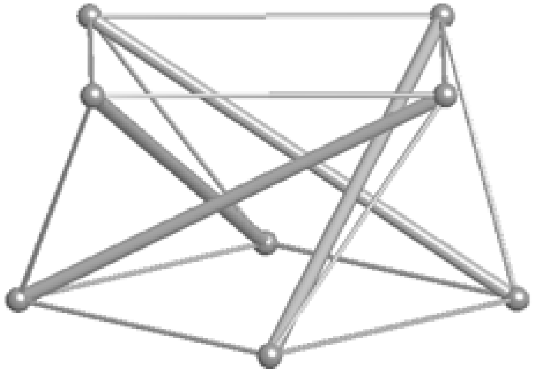
Update coordinates prestresses

$$\mathbf{Kd} = \mathbf{f}$$

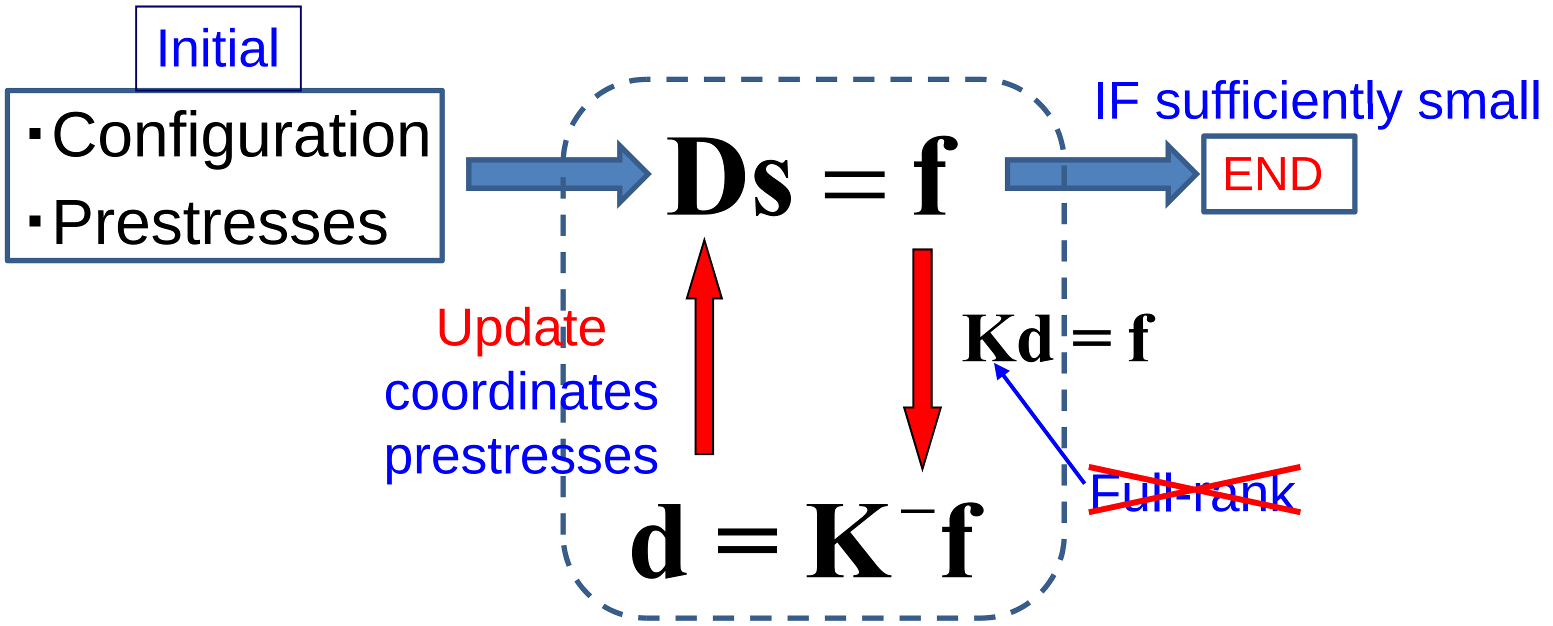
Full-rank

$$\mathbf{d} = \mathbf{K}^{-1}\mathbf{f}$$

NLA for Unstable Structures 不安定構造の非線形解析法



D: Equilibrium matrix
s: Prestress vector
f: Unbalanced force vector
K: Stiffness matrix
d: Nodal coordinates



Initial

- Configuration
- Prestresses

$$\mathbf{Ds} = \mathbf{f}$$

IF sufficiently small

END

Update coordinates prestresses

$$\mathbf{Kd} = \mathbf{f}$$

~~Full-rank~~

$$\mathbf{d} = \mathbf{K}^{-}\mathbf{f}$$

Generalized Inverse　一般逆行列

$$\mathbf{K} = \mathbf{\Psi} \begin{pmatrix} \lambda_1 & & & & & \\ & \ddots & & & \mathbf{O} & \\ & & \lambda_i & & & \\ & & & \lambda_{i+1} & & \\ & \mathbf{O} & & & \ddots & \\ & & & & & \lambda_n \end{pmatrix} \mathbf{\Psi}^{\mathrm{T}}$$

Singular value

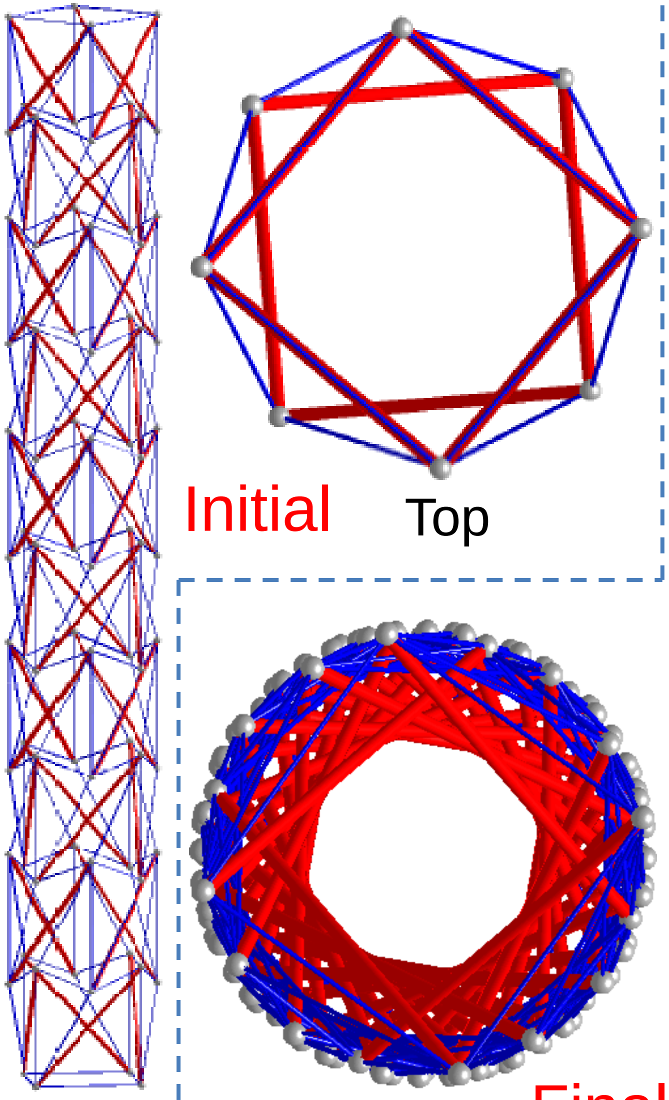
$$\lambda_i \neq 0 \longrightarrow 1/\lambda_i$$

$$\boxed{\lambda_i = 0 \longrightarrow 1/\lambda_i = 0}$$

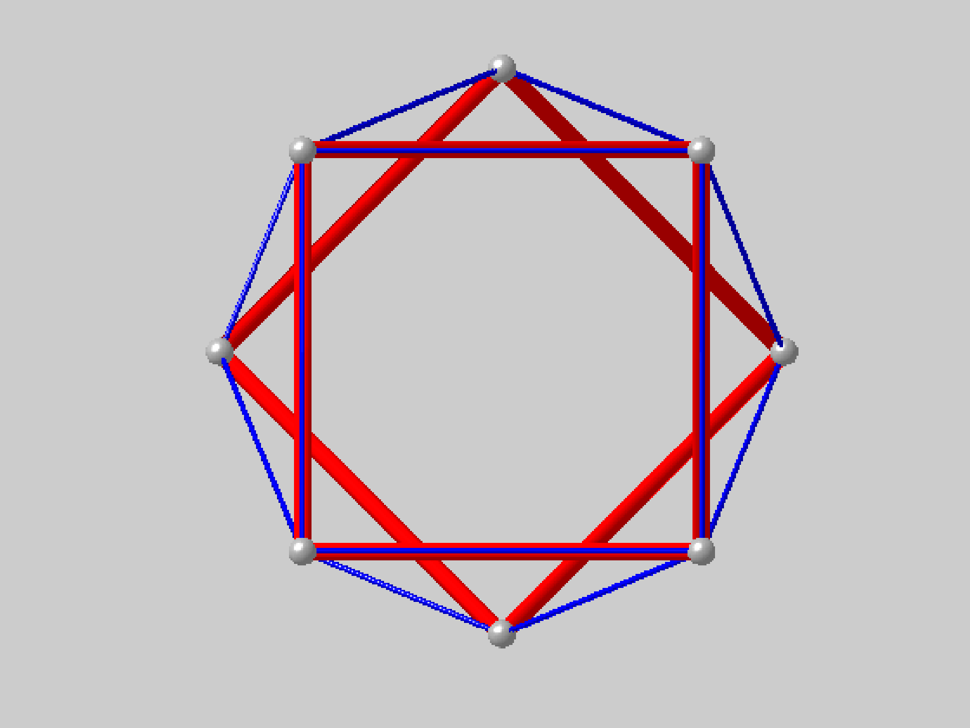
Rule out rigid-body motions

$$\mathbf{K}^{-} = \mathbf{\Psi} \begin{pmatrix} 1/\lambda_1 & & & & & \\ & \ddots & & & \mathbf{O} & \\ & & 1/\lambda_i & & & \\ & & & 1/\lambda_{i+1} & & \\ & \mathbf{O} & & & \ddots & \\ & & & & & 1/\lambda_n \end{pmatrix} \mathbf{\Psi}^{\mathrm{T}}$$

Tensegrity Tower by NLA

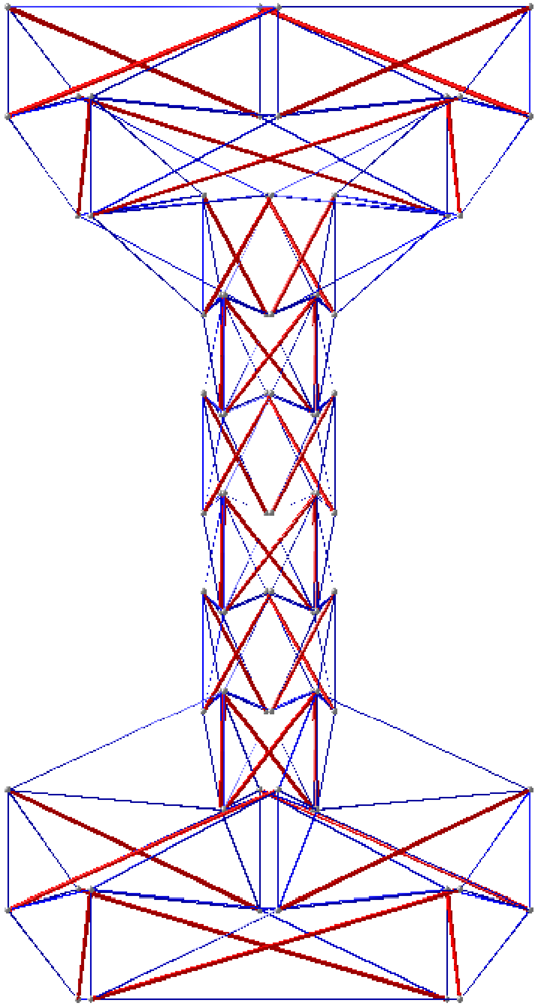


	Stiffness	Prestress
Struts	100	-1
Cables	10	1

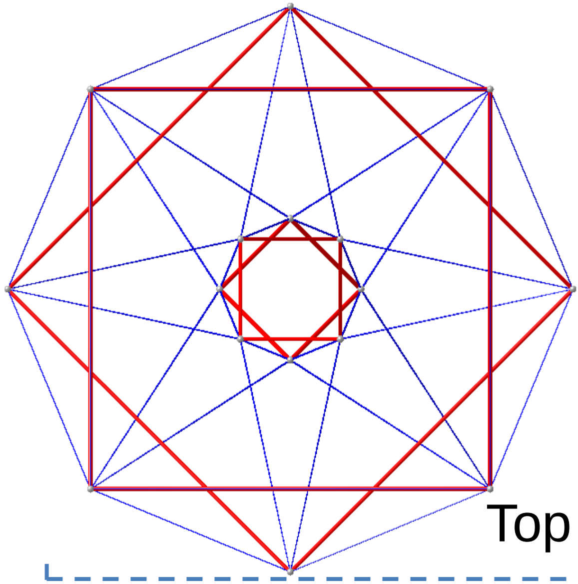


84 Iterations

Dumbbell Shape by NLA

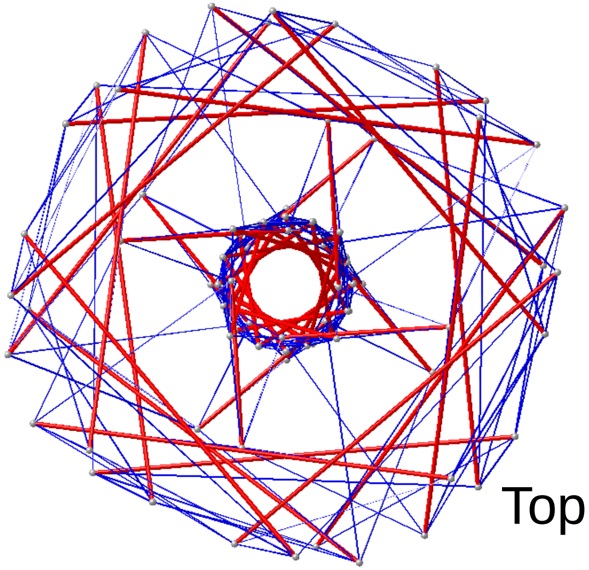


Side

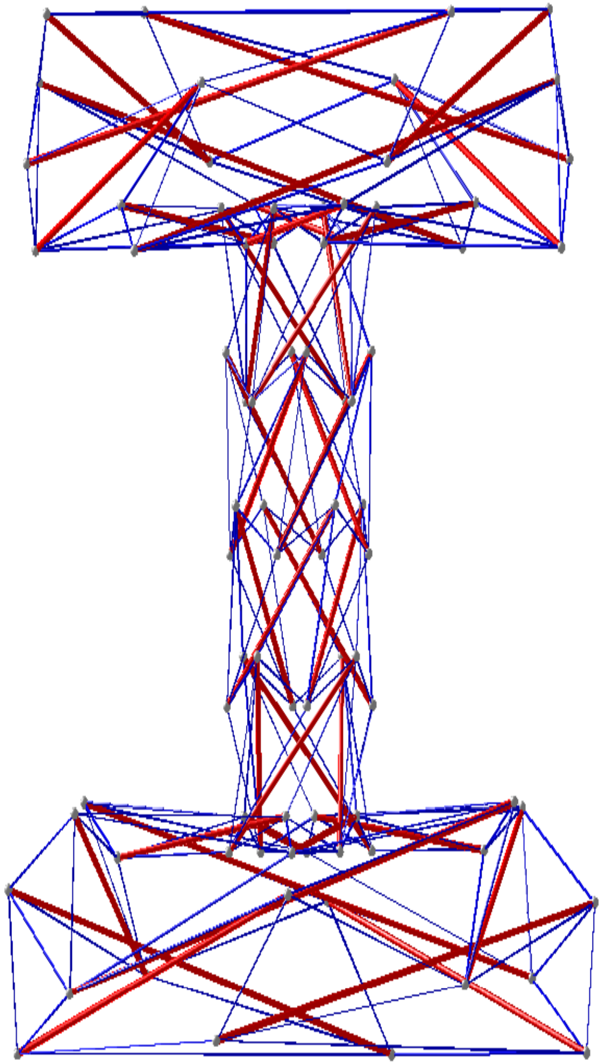


Top

Initial



Top

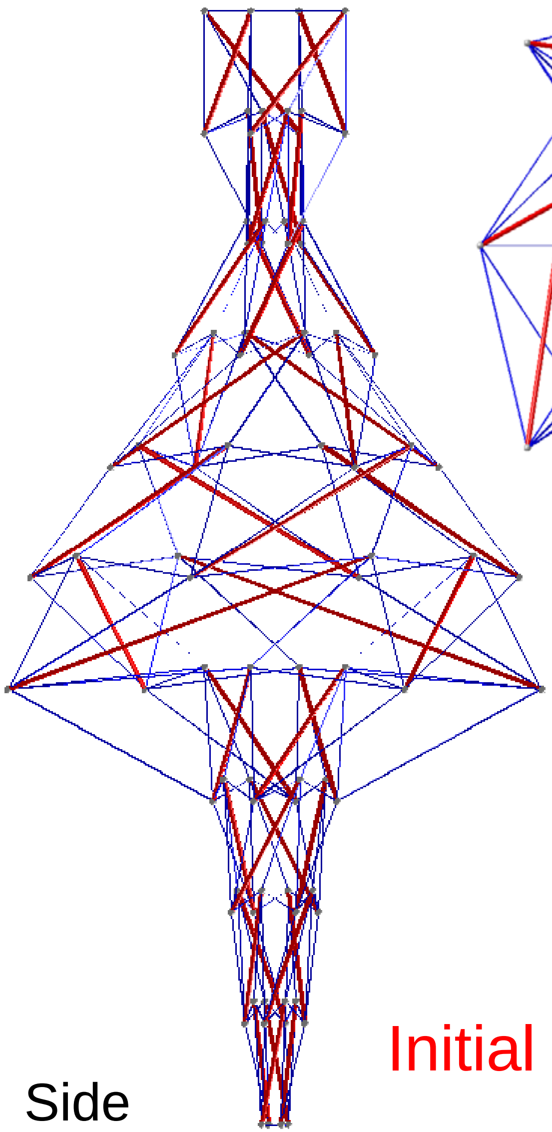


Side

Final

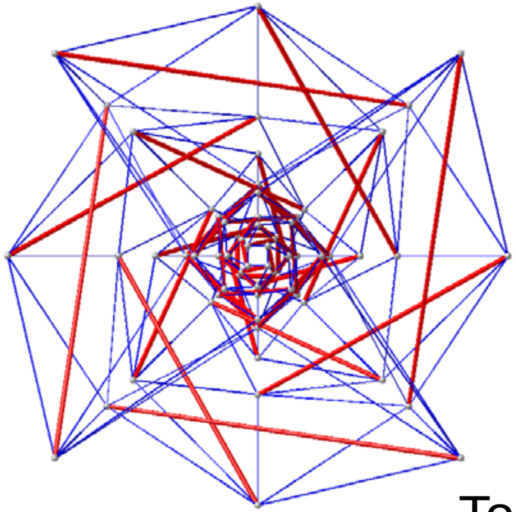
Video

Lady’s Mark by NLA

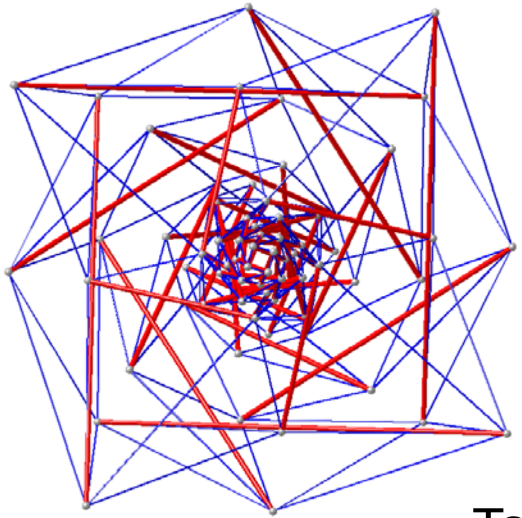


Side

Initial

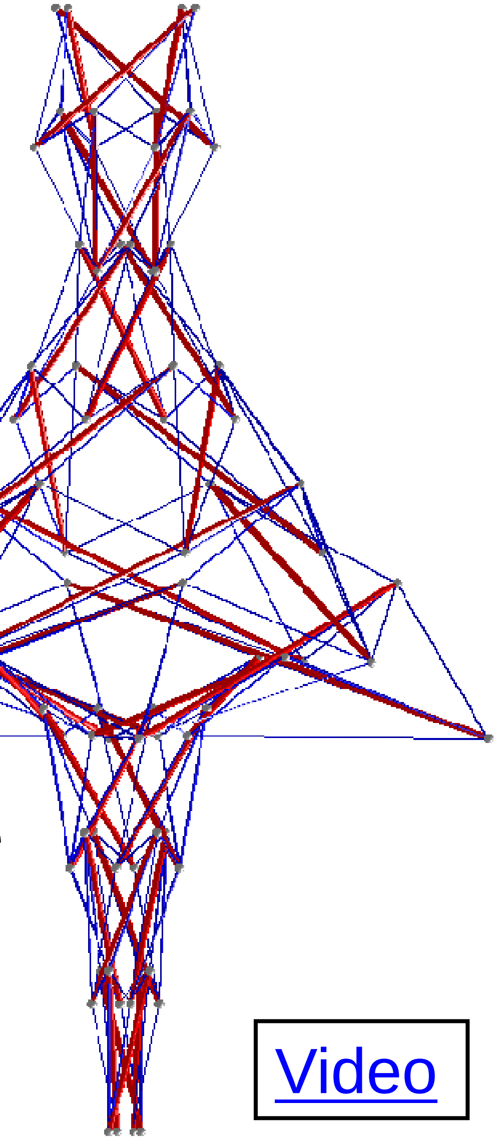


Top



Top

Final



Side

Video

Tensegrity Arch

Rainbow Arch, 2001
2.1 x 3.8 x 1m

Contents

- Introduction to Tensegrity
- Applications
- Stability
- Form-finding
  - Intuition Approaches
  - Analytical Approaches (using symmetry)
  - Numerical Approaches
    - Adaptive Force Density Method
    - Dynamic Relaxation Method
    - Non-linear Analysis Method
    - Optimization Method

Optimization Method
最適化手法

- <u>Optimization Problem</u>:

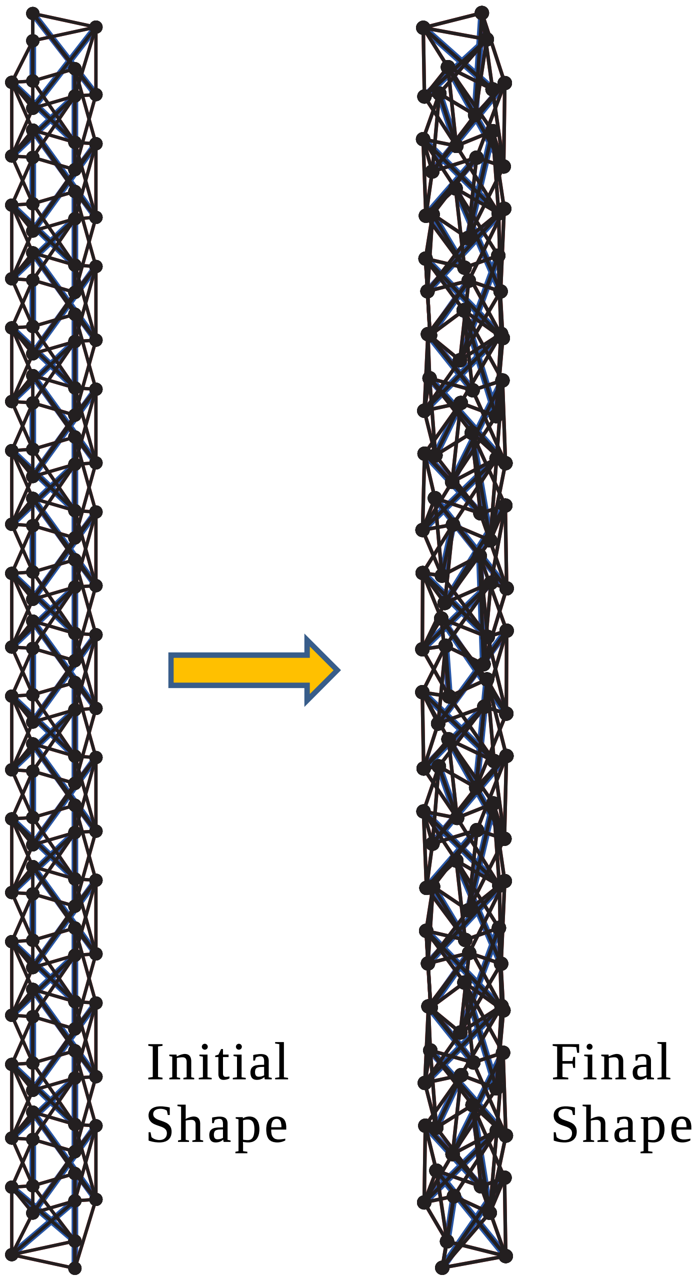
Minimize $$S(\boldsymbol{X}) = \sum_{i=1}^{m} S_i(L_i(\boldsymbol{X}))$$

Total Potential Energy

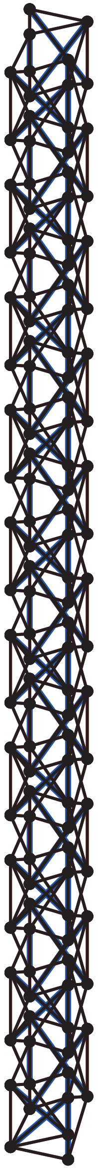
- <u>Stationary Condition</u>:

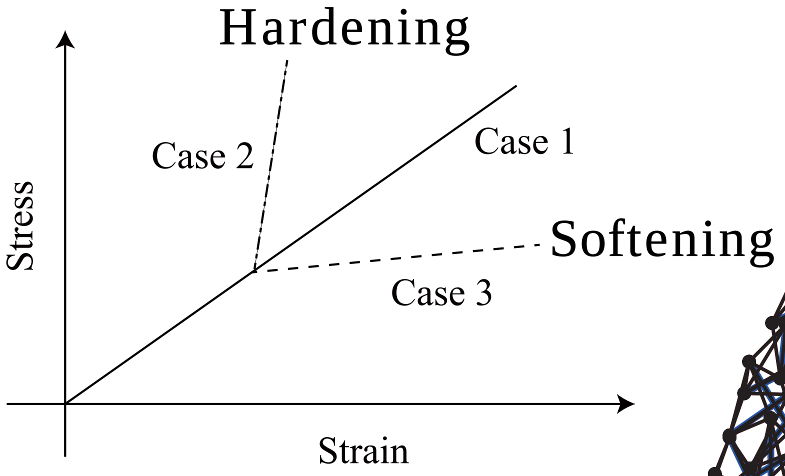
$$\frac{\partial S(\boldsymbol{X})}{\partial X_i} = \boxed{\sum_{i=1}^{m} \frac{\partial S_i(L_i(\boldsymbol{X}))}{\partial L_i}} \nabla L_i(\boldsymbol{X}) = \boldsymbol{0}$$

Member Force

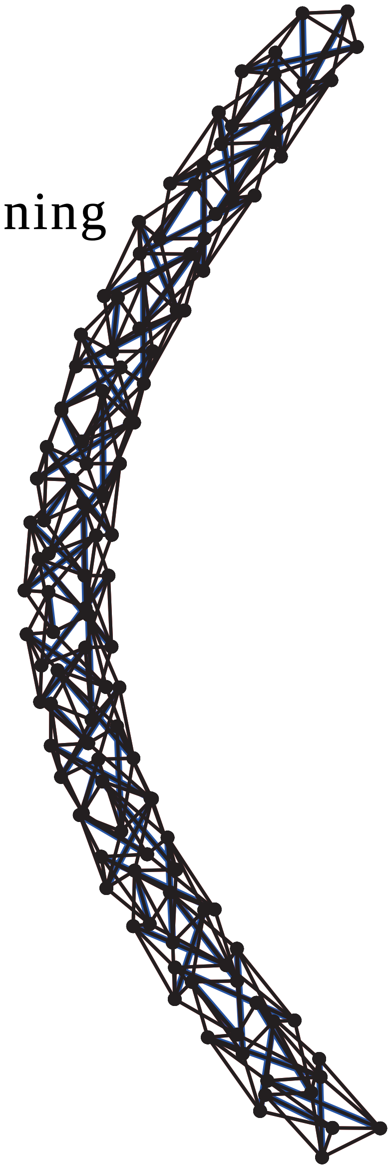


Optimization with Fictitious Material 仮想的材料性質

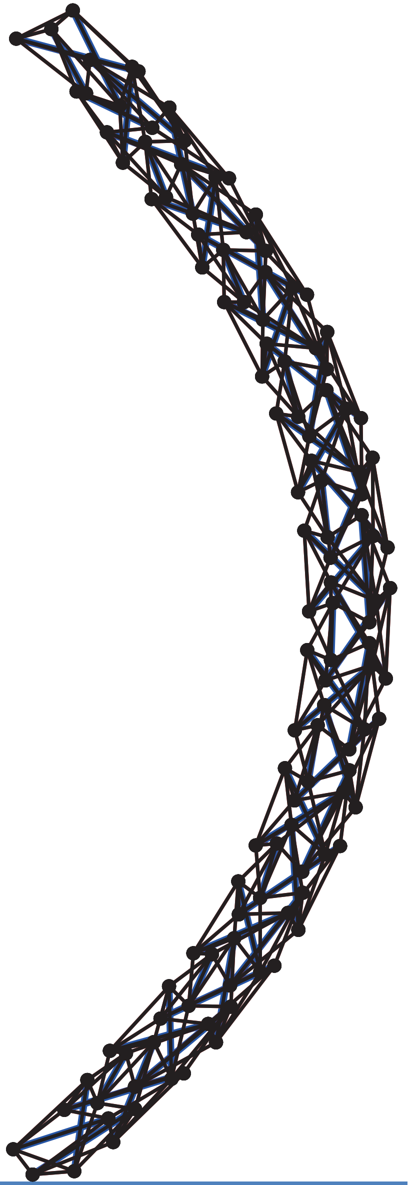




Initial Shape



OR



Final Shape

谢谢！

Thank you!

ありがとうございました！